A Ramsey-Theoretic Notion of Forcing

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The k-valued blocks, Fin_k

Definition

Let $k \in \omega \setminus \{0\}$ unless stated otherwise.

(1) For $a\colon \omega \to k+1$ we let $\mathrm{supp}(a) = \{n \in \omega : a(n) \neq 0\}$.

$$\mathrm{Fin}_k = \{a\colon \omega \to k+1 : \mathrm{supp}(a) \text{ is finite } \wedge k \in \mathrm{range}(a)\}.$$

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(2) $\mathrm{Fin}_{[1,k]} = \bigcup_{j=1}^{k} \mathrm{Fin}_j$.

(3) For $a, b \in \mathrm{Fin}_k$, we let $a < b$ denote $\mathrm{supp}(a) < \mathrm{supp}(b)$, i.e., $(\forall m \in \mathrm{supp}(a))(\forall n \in \mathrm{supp}(b))(m < n)$. A finite or infinite sequence $\langle a_i : i < m \leq \omega \rangle$ of elements of Fin_k is in block-position if for any $i < j < m$, $a_i < a_j$. The set $(\mathrm{Fin}_k)^\omega$ is the set of ω-sequences in block-position, also called block sequences.

Two operations on Fin_j

Definition

(4) For $k \geq 1$, $a, b \in \mathrm{Fin}_k$, we define the partial semigroup operation $+$ as follows: If $\mathrm{supp}(a) < \mathrm{supp}(b)$, then $a + b \in \mathrm{Fin}_k$ is defined. We let $(a + b)(n) = a(n) + b(n)$. Otherwise $a + b$ is undefined. Thus $a+b = a \upharpoonright \mathrm{supp}(a) \cup b \upharpoonright \mathrm{supp}(b) \cup 0 \upharpoonright (\omega \setminus (\mathrm{supp}(a) \cup \mathrm{supp}(b)))$.

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$a{+}b = a \restriction \mathrm{supp}(a) \cup b \restriction \mathrm{supp}(b) \cup 0 \restriction (\omega \setminus (\mathrm{supp}(a) \cup \mathrm{supp}(b)))$.

(5) For any $k \geq 2$ we define on Fin_k the Tetris operation: $T\colon \mathrm{Fin}_k \to \mathrm{Fin}_{k-1}$ by $T(a)(n) = \max\{a(n) - 1, 0\}$.

Definition

(6) Let $B \subseteq \mathrm{Fin}_k$ be min-unbounded, i.e., contain for any n some a with $\mathrm{supp}(a) > n$. We let

$$\begin{aligned}\mathrm{TFU}_k(B) =& \{T^{(j_0)}(b_{n_0}) + \cdots + T^{(j_\ell)}(b_{n_\ell}) \ : \\ & \ell \in \omega \setminus \{0\}, b_{n_i} \in B, b_{n_0} < \cdots < b_{n_\ell}, \\ & j_i \in k, \exists r \leq \ell j_r = 0\}\end{aligned}$$

be the partial subsemigroup of Fin_k generated by B. We call B a TFU_k-set if $B = \mathrm{TFU}_k(B)$.

The condensation order

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(8) We define the past-operation: Let $\bar{a} \in (\mathrm{Fin}_k)^\omega$ and $s \in \mathrm{Fin}_k$.

$$(\bar{a} \text{ past } s) = \langle a_i \; : \; i \geq i_0 \rangle$$

with $i_0 = \min\{i \; : \; \mathrm{supp}(a_i) > \mathrm{supp}(s)\}$.

A negation of near coherence for not necessarily centred families

Definition

1. Two subsets $\mathcal{F}_1$, $\mathcal{F}_2$ of $[\omega]^\omega$ are called nnc, not nearly coherent, if for any $X_i \in \mathcal{F}_i$, $i = 1, 2$ and any finite-to-one $h\colon \omega \to \omega$ there is $Y_i \subseteq X_i$, $Y_i \in \mathcal{F}_i$, $i = 1, 2$ such that $h[Y_1] \cap h[Y_2] = \emptyset$.

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2. Let $\mathcal{H} \subseteq (\mathrm{Fin}_k)^\omega$ and let $\mathcal{E}$ be a P-point. We say $\mathcal{H}$ avoids $\mathcal{E}$ if $\{\mathrm{supp}(\bar{a}) \ : \ \bar{a} \in \mathcal{H}\}$ is nnc to $\mathcal{E}$.

A subspace of $(\mathrm{Fin}_k)^\omega$–Fixing PP and $\bar{\mathcal{R}}$

Definition

We fix parameters as follows. Let $k \geq 1$. Fix $P_{\min}, P_{\max} \subseteq \{1, \ldots, k\}$. Let $PP = \{(i, x) : x \in \{\min, \max\}, i \in P_x\}$ and let

$$\bar{\mathcal{R}} = \{(\iota, \mathcal{R}_\iota) : \iota \in PP\}$$

be a PP-sequence of pairwise nnc Ramsey ultrafilters (pairwise nnc selective coideals, i.e., happy families, would suffice for the pure decision property and properness). We also name the end segments for $1 \leq j \leq k$:

$$\bar{\mathcal{R}} \restriction \{j, \ldots, k\} = \{(\iota, \mathcal{R}_\iota) : \iota = (i, x) \in PP \wedge i \in \{j, \ldots, k\}\}.$$

Happy families – selective coideals

Definition

We call $\mathcal{H} \subseteq [\omega]^\omega$ a selective coideal if

1. any cofinite subset of ω is in $\mathcal{H}$,
2. $\forall X \in \mathcal{H} \forall X_1, X_2 (X_1 \cup X_2 = X \rightarrow X_1 \in \mathcal{H} \vee X_2 \in \mathcal{H})$.
3. For any $\langle A_n : n < \omega \rangle$ such that for any n, $A_n \in \mathcal{H}$ and $A_{n+1} \subseteq A_n$ there is a diagonal lower bound $A \in \mathcal{H}$, i.e.,

$$\forall n \in A (A \setminus (n+1) \subseteq A_n).$$

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A Ramsey ultrafilter is a selective coideal that is also a filter.

A subspace of $(\mathrm{Fin}_k)^\omega$: The space $(\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$

Definition

We let $(\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$ denote the set of Fin_k-blocksequences $\bar{a}$ with the following properties:

- $(\forall i \in P_{\min})\{\min(a_n^{-1}[\{i\}]) \,:\, n \in \omega\} \in \mathcal{R}_{i,\min}$,
- $(\forall i \in P_{\max})\{\max(a_n^{-1}[\{i\}]) \,:\, n \in \omega\} \in \mathcal{R}_{i,\max}$,
-

$$(\forall s \in \mathrm{TFU}_k(\bar{a}))\big(\min(s^{-1}[\{1\}]) < \min(s^{-1}[\{2\}]) < \cdots < \min(s^{-1}[\{k-1\}] < \min(s^{-1}[\{k\}]) < \max(s^{-1}[\{k\}]) < \max(s^{-1}[\{k-1\}]) < \cdots < \max(s^{-1}[\{1\}])\big).$$

If $(i, x) \in \{1, \dots, k\} \times \{\min, \max\} \setminus PP$, we leave the term $x(s^{-1}[\{i\}])$ out of the order requirement.

We do not localise to a centred set

Lemma

There are $\sqsubseteq_k^*$-incompatible elements in $(\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$. Indeed, there are $\bar{a}$, $\bar{b} \in (\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$ such that for any $j = 0, \ldots, k-1$ the Fin_{k-j}-block-sequences $T^{(j)}[\bar{a}]$ and $T^{(j)}[\bar{b}]$ are $\sqsubseteq_{k-j}^*$-incompatible.

A common strengthening of a theorem by Gowers and a theorem by Blass

The special case of $PP = \{(1, \min), (1, \max)\}$ was proved by Blass in 1987, the case $PP = \emptyset$ and arbitrary finite k by Gowers in 1992.

Theorem

Let k, PP, $\bar{\mathcal{R}}$ be as above. Let $\bar{a} \in (\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$ and let c be a colouring of $\mathrm{TFU}_k(\bar{a})$ into finitely many colours. Then there is a $\bar{b} \sqsubseteq_k \bar{a}$, $\bar{b} \in (\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$, such that $\mathrm{TFU}_k(\bar{b})$ is c-monochromatic.

Diagonal lower bounds

Lemma

let k, PP, $\bar{\mathcal{R}}$ be as above. Any $\sqsubseteq_k$-descending sequence $\langle \bar{c}_n : n \in \omega \rangle$ in $(\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$ has a diagonal lower bound $\bar{b} \in (\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$

$$(\forall n \in \omega)((\bar{b} \text{ past } b_n) \sqsubseteq_k \bar{c}_{\max(\mathrm{supp}(b_n))+1}).$$

such that each b_{n+1} is an element of $\{c_{\ell_{n+1},m} : m \in \omega\}$ for some $\ell_{n+1} > \max(\mathrm{supp}(b_n))$ and b_0 is an element of $\{c_{\ell_0,m} : m \in \omega\}$ for some ℓ_0.

A k-stack of compact spaces

$\gamma(\mathrm{Fin}_j(\bar{\mathcal{R}} \restriction \{k+j-1, \ldots, k\}))$ is the set of ultrafilters $\mathcal{U}$ over Fin_j such that for any $\bar{a} \in \mathcal{U}$, $\ell \in \{1, \ldots, j\}$,

$$\{\min(a_n^{-1}[\{\ell\}]) \, : \, n \in \omega\} \in \mathcal{R}_{\ell+k-j,\min}$$

and analogously for $\max$.

Galvin–Glazer technique

Definition

For any $k \geq 1$, a reservoir of indices PP of the strict form is one of the following three types: $PP = \{(i, \min), (i, \max) : 1 \leq i \leq k\}$, $PP = \{(i, \min) : 1 \leq i \leq k\}$, $PP = \{(i, \min) : 1 \leq i \leq k\}$.

Definition and Lemma

Here we let PP be of the strict form. We define $\dot{+}$ on $(\bigcup_{j=1}^{k} \gamma(\mathrm{Fin}_j(\bar{\mathcal{R}} \restriction \{k-j+1, \ldots, k\})))^2$ as follows.

$$\begin{aligned} \dot{+} \colon & \gamma(\mathrm{Fin}_i(\bar{\mathcal{R}} \restriction \{k-i+1, \ldots, k\})) \times \gamma(\mathrm{Fin}_j(\bar{\mathcal{R}} \restriction \{k-j+1, \ldots, k\})) \\ & \to \gamma(\mathrm{Fin}_{\max\{i,j\}}(\bar{\mathcal{R}} \restriction \{k - \max(i,j) + 1, \ldots, k\})) \end{aligned}$$

is defined as

$$\begin{aligned} \mathcal{U} \dot{+} \mathcal{V} = \Big\{ & X \subseteq \mathrm{Fin}_{\max\{i,j\}}(\bar{\mathcal{R}} \restriction \{k - \max(i,j) + 1, \ldots, k\}) \\ & : \{s : \{t : s + t \in X\} \in \mathcal{V}\} \in \mathcal{U} \Big\}. \end{aligned}$$

A k-sequence of very good idempotent ultrafilters

Lemma

Still PP of the strict form. (Lemma 2.24, Todorcevic, Ramsey Spaces) Let k, PP, $\bar{\mathcal{R}}$ be as above, with full PP. For any $k \geq j \geq 1$, and $\bar{a} \in (\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$ there is an idempotent $\mathcal{U}_j \in \gamma(\mathrm{Fin}_j(\bar{\mathcal{R}} \restriction \{k+j-1, \ldots, k\}))$ such that for all $1 \leq i \leq j \leq k$

(1) $\mathcal{U}_j \dot{+} \mathcal{U}_i = \mathcal{U}_j$,

(2) $\dot{T}^{(j-i)}(\mathcal{U}_j) = \mathcal{U}_i$.

(3) $T^{(i-1)}(\bar{a}) \in \mathcal{U}_{k-i+1}$.

Stepping up to finite dimensions

Since the space $(\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$ is stable, we can step up the Milliken–Taylor style to higher finite arities:

Theorem

Let $n \in \omega \setminus \{0\}$ and $\bar{a} \in (\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$ and let c be a colouring of $(\mathrm{TFU}_k(\bar{a}))^n_<$ into finitely many colours. Then there is a $\bar{b} \sqsubseteq_k \bar{a}$, $\bar{b} \in (\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$ such that $(\mathrm{TFU}_k(\bar{b}))^n_<$ is c-monochromatic.

A useful notion of forcing

Definition

We let k, PP, $\bar{\mathcal{R}}$ be as above, not necessarily strict. In the Gowers–Matet forcing with $\bar{\mathcal{R}}$, $\mathbb{M}_k(\bar{\mathcal{R}})$, the conditions are pairs $(s, \bar{c})$ such that $s \in \mathrm{Fin}_k$ and $\bar{c} \in (\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$ and $\mathrm{supp}(s) < \mathrm{supp}(c_0)$.

The forcing order is: $(t, \bar{b}) \leq (s, \bar{a})$ if $t = s + s'$ and $s' \in \mathrm{TFU}_k(\bar{a})$ and $\bar{b} \sqsubseteq_k (\bar{a} \text{ past } s')$

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The forcing order is: $(t, \bar{b}) \leq (s, \bar{a})$ if $t = s + s'$ and $s' \in \mathrm{TFU}_k(\bar{a})$ and $\bar{b} \sqsubseteq_k (\bar{a} \, \mathrm{past} \; s')$

Lemma

$\mathbb{M}_k(\bar{\mathcal{R}})$ *has the pure decision property, i.e., for any* $\varphi \in \mathcal{L}(\in)$, $(s, \bar{a}) \in \mathbb{M}_k(\bar{\mathcal{R}}) \; \exists (s, \bar{b}) \leq (s, \bar{a}) \; ((s, \bar{b}) \Vdash \varphi \vee (s, \bar{b}) \Vdash \neg\varphi)$.

Good properties of the reservoirs of pure conditions

Definition

A set $\mathcal{H} \subseteq (\mathrm{Fin}_k)^\omega$ is called a Gowers–Matet-adequate family if the following hold:

1. $\mathcal{H}$ is closed $\sqsubseteq_k^*$-upwards.
2. $\mathcal{H}$ is stable, i.e., any $\sqsubseteq_k$-descending ω-sequence of members of $\mathcal{H}$ has a $\sqsubseteq^*$ lower bound in $\mathcal{H}$.
3. $\mathcal{H}$ has the Gowers property: If $\bar{a} \in \mathcal{H}$ and $\mathrm{TFU}_k(\bar{a})$ is partitioned into finitely many pieces then there is some $\bar{b} \sqsubseteq_k \bar{a}$, $\bar{b} \in \mathcal{H}$ such that $\mathrm{TFU}_k(\bar{b})$ is a subset of a single piece of the partition.

Examples of Gowers–Matet-adequate families

- $\mathcal{H} = (\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$, we write $\mathbb{M}(\bar{\mathcal{R}})$ for $\mathbb{M}(\mathcal{H})$.
- $\mathbb{M}(\mathcal{U})$ for a Gowers-Milliken-Taylor ultrafilter.
- Instead of imposing that $\min_i[\bar{a}]$, $\max_i[\bar{a}]$ come from happy families when $\bar{a} \in \mathcal{H}$ we could try to use $\mathrm{set}_i(\bar{a})$ for $i \in \{1, \ldots, k\}$.

The fate of the $\mathcal{R}_{i,x}$, $(i,x) \in PP$, in $\mathbf{V}^{\mathbb{M}(\bar{\mathcal{R}})}$

Definition

The i-fibre of the generic real $\mu = \bigcup\{s \upharpoonright \mathrm{supp}(s) : \exists \bar{a}(s, \bar{a}) \in G\}$ is

$$\mu_i = \bigcup\{s^{-1}[\{i\}] : \exists \bar{a}(s, \bar{a}) \in G\},$$

$\mathrm{supp}(\mu)$ is the union of the μ_i.

Density argument: μ_i is not measured by $\mathcal{R}_{i,\min}$, $\mathcal{R}_{i,\max}$.

Definition

Let $X \in [\omega]^\omega$. We let $f_X(n) = |X \cap n|$.

The fate of other ultrafilters in $\mathbf{V}^{\mathbb{M}(\bar{\mathcal{R}})}$

Lemma

Let $h\colon \omega \to \omega$ be a finite-to-one function. Let $\mathcal{E}$ and $\mathcal{W}$ be ultrafilters over ω such that $\mathcal{W}$, $\mathcal{E} \not\geq_{RB} \mathcal{R}_\iota$ for $\iota \in PP$. Then for any $(s, \bar{a}) \in \mathbb{M}_k(\bar{\mathcal{R}})$, $E \in \mathcal{E}$ there are $\bar{b} \sqsubseteq_k \bar{a}$, $\bar{b} \in (\mathrm{Fin}_k)^\omega(\bar{\mathcal{R}})$, and $E' \in \mathcal{E}$, $E' \subseteq E$ and $W \in \mathcal{W}$ such that

(1) $h[\bigcup\{\mathrm{supp}(b_n) \,:\, n \in \omega\}] \cap h[E'] = \emptyset$.

(2) $h[\bigcup\{[\min(\mathrm{supp}(b_n)), \max(\mathrm{supp}(b_n))] \,:\, n \in \omega\}]$
$\cap (h[E'] \cup h[W]) = \emptyset$, *and*
$(s, \bar{b}) \Vdash_{\mathbb{M}_k(\bar{\mathcal{R}})} f_{\mathrm{supp}(\mu)}[W] = f_{\mathrm{supp}(\mu)}[E']$.

Fruit of conclusion 1 of the lemma

Theorem

(Adaption of a theorem of Eisworth) Let $k \geq 1$ *and* $\bar{\mathcal{R}}$ *be as above and assume that* $\mathcal{E}$ *is a* P*-point with* $\mathcal{E} \not\geq_{RB} \mathcal{R}_{(i,\min)}, \mathcal{R}_{(j,\max)}$ *for any* $i \in P_{\min}$ *and* $j \in P_{\max}$*. Then* $\mathcal{E}$ *continues to generate an ultrafilter after we force with* $\mathbb{M}_k(\bar{\mathcal{R}})$*.*

Fruit of conclusion 2 of the lemma

Theorem

Let $k \geq 1$ and $\bar{\mathcal{R}}$ be as above and assume $\mathcal{E}, \mathcal{W} \not\geq_{RB} \mathcal{R}_{(i,\min)}, \mathcal{R}_{(j,\max)}$ for any $i \in P_{\min}$ and $j \in P_{\max}$ and let $\mathcal{E}$ be a P-point and $\mathcal{W}$ be an ultrafilter over ω. Then

$$\mathbb{M}_k(\bar{\mathcal{R}}) \Vdash f_{\mathrm{supp}(\mu)}(\mathcal{E}) = f_{\mathrm{supp}(\mu)}(\mathcal{W}).$$

Start of an (cs) iteration

Now we are concerned with the second iterand. The following follows from an easy density argument.

Lemma

Let $\iota = (i, x) \in PP$.

$$\mathbb{M}_k(\bar{\mathcal{R}}) \Vdash \mathcal{R}_\iota \cup \{\mu_i\} \text{ is a filter subbase.}$$

Finding a second iterand

Theorem

Let k, PP, $\bar{\mathcal{R}}$ be as in the non-strict form, $\iota \in PP$.

$$\mathbb{M}_k(\bar{\mathcal{R}}) \Vdash (\text{filter}((\mathcal{R}_\iota \cup \{\mu_i\}))^+ \textit{ is a happy family that avoids } \mathcal{E}$$
$$\textit{and for } \iota \neq \iota' \textit{ the family } (\text{filter}((\mathcal{R}_\iota \cup \{\mu_i\}))^+$$
$$\textit{is nnc to the family } (\text{filter}((\mathcal{R}_{\iota'} \cup \{\mu_{i'}\}))^+.$$

and hence

$$\mathbb{M}_k(\bar{\mathcal{R}}) \Vdash (\exists \mathcal{R}_\iota^{\text{ext}} \supseteq (\mathcal{R}_\iota \cup \{\mu_i\})\big(\mathcal{R}_\iota^{\text{ext}} \textit{ is a Ramsey ultrafilter}$$
$$\textit{that is nnc to } \mathcal{E} \textit{ and for } \iota \neq \iota', \mathcal{R}_\iota^{\text{ext}} \textit{ nnc } \mathcal{R}_{\iota'}^{\text{ext}}\big).$$

Lemma

(Existence of positive diagonal lower bounds) Let $\mathcal{U}$ *be an Milliken–Taylor ultrafilter,* $\mathcal{E}$ *be a* P*-point,* $\Phi(\mathcal{U}) \not\leq_{\mathrm{RB}} \mathcal{E}$*. Let* $\mathbb{Q} = \mathbb{M}(\mathcal{U})$ *and let* μ *be the name for the generic real. Let* $\utilde{\bar{X}} = \langle \utilde{X}_n : n \in \omega \rangle$ *be a sequence of* $\mathbb{Q}$*-names for elements of* $(\mathrm{Fin})^\omega$ *such that*

$$\mathbb{Q} \Vdash (\forall n \in \omega)(\utilde{X}_n \in (\mathcal{U} \upharpoonright \mu)^+ \wedge \utilde{X}_{n+1} \sqsubseteq \utilde{X}_n).$$

Continued

Lemma continued

Then

$$\underset{\sim}{D} = \Big\{\langle t,(s,\bar{a})\rangle \,:\, (s,\bar{a}) \in \mathbb{Q} \wedge (\exists k \in \omega)(\exists t_0 < t_1 < \cdots < t_{k-1} \in [\mathrm{Fin}]^k_<)$$

$$\Big(t_{k-1} < t_k = t \wedge (s,\bar{a}) \Vdash \text{``}t_0 = \min_{\mathrm{Fin}}(\underset{\sim}{X}_0 \restriction \mu) \wedge$$

$$\bigwedge_{i<k} t_{i+1} = \min_{\mathrm{Fin}}((\underset{\sim}{X}_{\max(t_i)+1} \restriction \mu) \text{ past } t_i)\text{''}\Big)\Big\}$$

fulfils

$$\mathbb{Q} \Vdash \underset{\sim}{D} \in (\mathcal{U} \restriction \mu)^+ \wedge \underset{\sim}{D} \sqsubseteq \underset{\sim}{X}_0 \wedge (\forall t \in \underset{\sim}{D})((\underset{\sim}{D} \text{ past } t) \sqsubseteq \underset{\sim}{X}_{\max(t)+1}).$$

A ZFC result

Proposition

Let $\mathcal{E}$ be a filter over ω, and let $\mathcal{V}$ and $\mathcal{W}$ be two filters over ω that are not nearly coherent to $\mathcal{E}$. If $\mathcal{V}$ is nearly coherent to $\mathcal{W}$, then there is $E \in \mathcal{E}$ such that $f_E(\mathcal{V}) \cup f_E(\mathcal{W})$ is a filter subbase.

Carrying on in cs limit steps

Theorem

Suppose that $\mathbb{P}_\beta, \bar{\mathcal{R}}_\beta$ *are as above* $\mathbb{P}_\alpha$ *is the countable support limit of* $\langle \mathbb{P}_\beta, \mathbb{M}_k(\bar{\mathcal{R}}_\beta) : \beta < \alpha \rangle$. *In* $\mathbf{V}^{\mathbb{P}_\alpha}$, *for any* $\iota \in PP$, *the set of positive sets*

$$\Big(\bigcup_{\gamma<\alpha} (\mathcal{R}_{\gamma,\iota} \cup \{\mu_{\gamma,i}\}) \Big)^+$$

forms a happy family that avoids $\mathcal{E}$ *and the happy families are pairwise nnc.*

Near coherence classes in an iteration of length ω_2

Theorem

Let $\mathcal{E}$ be a P-point and assume CH and let $k \geq 1$ and let $PP \subseteq \{(i,x) \,:\, x = \min, \max, i = 1, \dots, k\}$. Then there is a countable support iteration iteration of proper iterands $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{M}_k(\bar{\mathcal{R}}_\beta) \,:\, \beta < \omega_2, \alpha \leq \omega_2 \rangle$ that in the extension there exactly $|PP| + 1$ near-coherence classes of ultrafilters. Namely, one class is represented by a P-point of character ω_1 and $|PP|$ classes represented by the Ramsey ultrafilters

$$\mathcal{R}_{i,x} = \bigcup \{\mathcal{R}_{i,x,\alpha} \,:\, \alpha < \omega_2\},$$

$(i,x) \in PP$.

A factorisation

Proposition

We let $\mathbb{Q}_{\mathrm{pure}} = (\mathrm{Fin}^{\omega}_{k}(\bar{\mathcal{R}}), \sqsubseteq^{*}_{k})$ *and we let* $\underset{\sim}{\mathcal{U}} = \{\langle \bar{a}, \check{\bar{a}} \rangle : \bar{a} \in \mathbb{Q}_{\mathrm{pure}}\}$*. Then the following holds:*

(1) $\mathbb{Q}_{\mathrm{pure}}$ *is* ω*-closed.*

(2) $\mathbb{M}_k(\bar{\mathcal{R}})$ *is densely embedded into* $\mathbb{Q}_{\mathrm{pure}} * \mathbb{M}_k(\underset{\sim}{\mathcal{U}})$*.*

(3) $\mathbb{Q}_{\mathrm{pure}}$ *forces that* $\underset{\sim}{\mathcal{U}}$ *is a Gowers–Milliken–Taylor ultrafilter with* $\hat{\min}_i(\mathcal{U}) = \mathcal{R}_{i,\min}$ *and* $\hat{\max}_j(\mathcal{U}) = \mathcal{R}_{j,\max}$*.*

(4) $\mathbb{Q}_{\mathrm{pure}}$ *forces that* $\Phi(\underset{\sim}{\mathcal{U}})$ *is nnc to any filter from the ground model that is nnc* $\mathcal{R}_{\iota}$*,* $\iota \in PP$*.*

Proof of a conjecture of Blass

In 1987 Blass conjectured that the existence of two non-isomorphic Ramsey ultrafilters does not imply the existence of a Milliken–Taylor ultrafilter.

Theorem

For any k, PP, $\bar{\mathcal{R}}$ in the forcing extensions from the main theorem, is there is no Gowers–Milliken–Taylor ultrafilter over $\mathrm{Fin}_{k'}$ for any $k' \geq 1$.

Reason: If $\mathcal{V}$ is an Milliken–Taylor ultrafilter, then this holds for $\mathbb{P}_\alpha$-part in $\mathbf{V}^{\mathbb{P}_\alpha}$ for club many $\alpha < \omega_2$. Under CH, the core $\Phi(\mathcal{V}) \cap \mathbf{V}^{\mathbb{P}_\alpha}$ contains a tree of 2^{ω_1} near coherence classes.