A Ramsey Theorem for Metric Spaces

("joint work" with Saharon Shelah)

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Classical Ramsey Theory

$$\kappa \to (\lambda)^{\nu}_{\mu}$$

- For any coloring of ν-sized subsets of κ with μ-many colors there is a λ-sized monochromatic subset of κ.
- Case $\nu = 1$ is trivial.
- What if we add structure?

Adding Structure

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Ramsey arrow with structure

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Ordered Graphs

Let $\mathcal{G}$ be the class of well-ordered undirected graphs and $\mathrm{i} \in \mathrm{Emb}(\mathrm{G}, \mathrm{H})$ if i is an order preserving injective mapping and the image of G is an induced subgraph of H graph-isomorphic to G.

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Theorem (Hajnal,Komjáth)

$$2^\kappa \rightarrow_{\mathcal{G}} (\kappa)^1_\kappa.$$

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Topological Spaces

Let $\mathcal{T}_0$ and $\mathcal{T}_1$ be the class all T_0 and T_1 topological spaces, respectively. The set $\mathrm{Emb}(X, Y)$ consists of homeomorphic embeddings of a topological space X into Y.

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Theorem (Weiss) Every T_2 topological space can be partitioned into two pieces such that no piece contains a homeomorphic copy of the Cantor set (under suitable cardinal arithmetic assumptions; these hold e.g. if no measurable cardinals exist).

Metric spaces

Let $\mathcal{M}$ be the class of metric spaces. For what κ, λ, μ can we have

$$\kappa \to_{\mathcal{M}} (\kappa)^1_\mu.$$

- By Weiss' result we need $\lambda < 2^\omega$
- We restrict ourselves to bounded metric spaces.
- The embeddings will be scaled isometries

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Definition A metric space (X, ρ) is **bounded** if there is d such that $\rho(x, y) < d$ for each (X, ρ).

Definition An injective map $i : (X, \rho) \to (Y, \sigma)$ is a **scaled isometry** if there is $\varepsilon > 0$ such that for each $x, y \in X$ we have $\rho(x, y) = \varepsilon \cdot \sigma(i(x), i(y))$.

Let $\mathcal{M}$ consist of all bounded metric spaces and $\mathrm{Emb}(X, Y)$ be the scaled isometries of X into Y.

Theorem (Shelah, V.)

$$2^\omega \to_{\mathcal{M}} (\omega)^1_\omega.$$

Given a countable bounded metric space (X, ρ) we will find a metric space (Y, d) such that for any partition of Y into countably many pieces one piece contains a scaled isometric copy of X.

- By universality of $\mathbb{Q} \cap [0, 1]$ we could only consider $X = \mathbb{Q} \cap [0, 1]$; this will not be needed.

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The underlying set

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Let

$$Y = \{s \in {}^{<\omega_1}\omega : (\forall r)(|\mathrm{Pr}(s, r)| < \omega)\}.$$

Given $s \in Y$ and r in the range of $\pi(s)$ we let

$$X(r, s) = \{s \upharpoonright \alpha : s(\alpha) = r\}.$$

The (pseudo)metric

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- define a function f on $X(r, s) \times X(r, s) \subseteq Y \times Y$ so that it is a suitably scaled isometric copy of an initial segment of X.

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Then

- γ is a limit ordinal and $s_\gamma \notin Y$
- $X(r, s)$ has order typy ω for some r.
- $X(r, s) \subseteq Y$ is a scaled r-colored copy of X.

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