Definable discrete sets, Ramsey theory and forcing

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Galvin's theorem

Theorem (Galvin, 1968)

Suppose X is a non-empty perfect Polish space and

$$c\colon X^2 \to \{0,1\}$$

is symmetric and Baire measurable.
Then there is a perfect set $C \subseteq X$ such that c is constant on

$$C^2 \setminus \mathrm{diag}$$

Perfect trees

Sacks forcing is the set of *perfect trees* $p \subseteq 2^{<\omega}$, ordered by inclusion.

The *branch set* of p is

$$[p] = \{x \in 2^\omega \mid (\forall k \in \omega)\ x \restriction k \in p\}$$

Let $s \in 2^{<\omega}$.

$$p_s = \{t \in p \mid t \subseteq s \vee s \subseteq t\}.$$

is a perfect tree *iff* $s \in p$.

p_n is the n-th splitting level of p.

Galvin's Theorem phrased for Sacks forcing

We can phrase Galvin's Theorem in terms of Sacks forcing.

Theorem (Galvin's Theorem, equivalent form)

Let $p \in \mathbb{S}$ *and*

$$c\colon [p]^2 \to \{0, 1\}$$

be symmetric and Baire measurable.
Then there is $q \in \mathbb{S}$, $q \leq p$ *such that* c *is constant on*

$$[q]^2 \setminus \mathrm{diag}$$

Question

Is there an analogue for *iterated* Sacks forcing?

- Let $\mathbb{P}$ be an iteration of Sacks forcing with countable support, of length λ.
- For $\xi \leq \lambda$, denote by $\mathbb{P}_\xi$ the initial segment of $\mathbb{P}$.
- Recall that $\mathbb{P}$ consists of sequences $\bar{p} \colon \lambda \to V$ such that
  1. For each $\xi < \lambda$, $\bar{p}(\xi)$ is a $\mathbb{P}_\xi$-name for a perfect tree.
  2. $\mathrm{supp}(\bar{p})$ is countable, where

$$\mathrm{supp}(\bar{p}) = \{\xi < \lambda \mid \bar{p} \restriction \xi \not\Vdash \bar{p}(\xi) = 2^{<\omega}\}$$

- $\mathbb{P}$ adds a sequence of length λ,

$$\bar{s}_G \in (2^\omega)^\lambda$$

  such that $\bar{s}_G(\xi)$ is Sacks over $V[\bar{s}_G \restriction \xi]$.

Galvin's Theorem for iterated Sacks forcing?

- Let $\bar{p} \in \mathbb{P}$. What is $[\bar{p}]$?
- Provided we can define $[\bar{p}]$...

Question:

Is there for every $\bar{p} \in \mathbb{P}$ and every

$$c\colon [\bar{p}]^2 \to \{0, 1\}$$

which is symmetric and *nice*, some $\bar{q} \in \mathbb{P}$, $\bar{q} \leq \bar{p}$ such that c is constant on $[\bar{q}]^2 \setminus \text{diag}$?

What do I mean by *nice*?

- Answer is 'yes' for *continuous c* (Geschke-Kojman-Kubiś-Schipperus)
- perhaps Baire measurable...?

What is $[\bar{p}]$?

Let $\bar{p}$ be an iterated Sacks condition.
Let $\bar{t}\colon \lambda \to 2^{<\omega}$ be finitely supported, i.e.

$$\{\xi \mid \bar{t}(\xi) \neq \emptyset\} \text{ is finite.}$$

What is $\bar{p}_{\bar{t}}$?

Definition

1. Define $\bar{p}_t$ by induction as the sequence of names $\bar{q}$ such that for each $\xi < \lambda$,

$$\bar{q} \upharpoonright \xi \in \mathbb{P}_\xi \Rightarrow \bar{q} \upharpoonright \xi \Vdash_{\mathbb{P}_\xi} \bar{q}(\xi) = \bar{p}(\xi)_{t(\xi)}$$

2. We say $\bar{p}$ *accepts* $\bar{t}$ **iff** $\bar{p}_t \in \mathbb{P}$.

Note that $\bar{p}$ accepts $\bar{t}$ **iff** at every step we have

$$\bar{p}_t \upharpoonright \xi \Vdash \bar{t}(\xi) \in p(\xi).$$

For a dense set of $\bar{p} \in \mathbb{P}$ we have:

0 There is $F_0 \colon \bar{p}(0) \to$ FINITE TREES and $\sigma_1 \in \operatorname{supp}(\bar{p})$ such that

$$(\forall n \in \omega)(\forall t \in p(0)_n)\ (\bar{p} \restriction \sigma_1)_t \Vdash_{\mathbb{P}_{\sigma_1}} F_0(t) = \bar{p}(\sigma_1)_n$$

1 There is a function F_1 and $\sigma_2 \in \operatorname{supp}(\bar{p})$ such that (letting $\sigma_0 = 0$)

$$(\forall n \in \omega)(\forall \bar{t} \colon \{\sigma_0, \sigma_1\} \to 2^{<\omega}) \\ (\bar{t}(0) \in p(0)_n \wedge \bar{t}(1) \in F_0(t_0)) \Rightarrow \\ (\bar{p} \restriction \sigma_2)_{\bar{t}} \Vdash_{\mathbb{P}_{\sigma_1}} F_1(\bar{t}) = \bar{p}(\sigma_2)_n$$

ω And so on: There exists sequences $F_0, \dots, F_k, \dots$ and $\sigma_0, \dots, \sigma_k, \dots$ with $\sigma_0 = 0$ such that the analogous holds for each $k \in \omega$ and

$$\{\sigma_k \mid k \in \omega\} = \operatorname{supp}(\bar{p})$$

Fix $\bar{p}$ and $F_0, F_1, \ldots$ as in the previous slide.

Define a partial function

$$F_k^* \colon (2^\omega)^{\{\sigma_0,\ldots,\sigma_k\}} \to \text{PERFECT TREES}$$

by

$$F_k^*(\bar{x}) = \bigcup_{n\in\omega} F_k(\bar{x} \restriction n)$$

Then $[\bar{p}]$ is the subspace of $(2^\omega)^\lambda$ consisting of

$$\bar{x} \colon \operatorname{supp}(\bar{p}) \to 2^\omega$$

such that for each $n \in \omega$

$$\bar{x}(n) \in F_n^*(\bar{x} \restriction n)$$

A counterexample

Let $\bar{p} \in \mathbb{P}$. Fix $\xi < \lambda$.

Define a symmetric Borel function

$$c\colon [\bar{p}]^2 \to \{0, 1\}$$

by

$$c(\bar{x}_0, \bar{x}_1) = \begin{cases} 1 & \text{if } \bar{x}_0(\xi) \neq \bar{x}_1(\xi) \\ 0 & \text{otherwise} \end{cases}$$

Note:

- Every $\bar{q} \leq \bar{p}$ will meet both colours
- $c^{-1}(1)$ is open, $c^{-1}(0)$ is closed.

For $\bar{x}_0, \bar{x}_1 \in [\bar{p}]$, let

$$\Delta(\bar{x}_0, \bar{x}_1) = \text{the least } \xi \text{ such that } \bar{x}_0(\xi) \neq \bar{x}_1(\xi).$$

Let

$$\Delta_\xi = \{(\bar{x}_0, \bar{x}_1) \in [\bar{p}]^2 \mid \Delta(\bar{x}_0, \bar{x}_1) = \xi\}$$

Question:

Can we show: For every $\bar{p} \in \mathbb{P}$ and for evey *nice* symmetric c,

$$c\colon [\bar{p}]^2 \to \{0, 1\}$$

there is $\bar{q} \in \mathbb{P}$, $\bar{q} \leq \bar{p}$ such that c only depends on $\Delta(\cdot, \cdot)$ on $[\bar{q}]^2 \setminus \text{diag}$?

Try again...

Let me restate the previous question:

Question:

Can we show: For every $\bar{p} \in \mathbb{P}$ and for evey *nice* symmetric

$$c\colon [\bar{p}]^2 \to \{0, 1\}$$

there is $\bar{q} \in \mathbb{P}$, $\bar{q} \leq \bar{p}$ such that c is constant on $\Delta_\xi \cap [\bar{q}]^2 \setminus \text{diag}$?

- Δ_0 is comeager in $[\bar{p}]^2$
- So *nice* must be more restrictive than Baire measurable!
- otherwise: take c arbitrary on Δ_ξ, $\xi > 0$ (a meager set!)

. . . fail again

Another counterexample:
Fix a bijection $G \colon \operatorname{supp}(\bar{p}) \setminus \{0\} \to \omega$.
Define a symmetric function

$$c \colon [\bar{p}]^2 \to \{0, 1\}$$

as follows. Given $(\bar{x}_0, \bar{x}_1)$, let $\bar{x}_i$ be such that for $\xi = \Delta(\bar{x}_0, \bar{x}_1)$

$$\bar{x}_i(\xi) <_{\text{lex}} \bar{x}_{1-i}(\xi)$$

If $\xi \in \operatorname{supp}(\bar{p})$ and $G(\xi) = k$, set

$$c(\bar{x}_0, \bar{x}_1) = \bar{x}_i(0)(k).$$

(When $\xi \in \operatorname{supp}(\bar{p})$ fails, set c to be 0; this case is irrelevant)

Suppose $\bar{q} \in \mathbb{P}$ is such that

$$(\forall \xi \in \mathrm{supp}(\bar{q}))\ c \text{ has constant value } I(\xi) \text{ on } \Delta_\xi \cap [\bar{q}]^2 \setminus \text{diag}.$$

We reach a contradiction:

- pick $\bar{x}_0$ as follows:
  1. $\bar{x}_0(0)$ is **arbitrary** in $[\bar{q}(0)]$
  2. $\bar{x}_0(\xi)$ for $\xi > 0$ always picks the left-most branch
- For every $\xi > 0$, we can pick $\bar{x}_1^\xi$ such that
  1. $\Delta(\bar{x}_0, \bar{x}_1^\xi) = \xi$,
  2. $\bar{x}_1^\xi(\xi)$ is lexicographically after $\bar{x}_0(\xi)$
- Thus, for each $\xi \in \mathrm{supp}(\bar{p})$,

$$\bar{x}_0(0)(G(\xi)) = c(\bar{x}_0, \bar{x}_1^\xi) = I(\xi),$$

  completely determining $\bar{x}(0)$; contradiction.

The solution:

Theorem (Galvin's Theorem for iterated Sacks forcing)

For every $\bar{p} \in \mathbb{P}$ and every symmetric universally Baire

$$c \colon [\bar{p}]^2 \to \{0, 1\}$$

there is $\bar{q} \in \mathbb{P}$, $\bar{q} \leq \bar{p}$, with an enumeration $\langle \sigma_k \mid k \in \omega \rangle$ of $\operatorname{supp}(\bar{q})$ and a function $N \colon \operatorname{supp}(\bar{q}) \to \omega$ such that for $(\bar{x}_0, \bar{x}_1) \in [\bar{q}]^2 \setminus \mathrm{diag}$, the value of $c(\bar{x}_0, \bar{x}_1)$ only depends on

$$\xi = \Delta(\bar{x}_0, \bar{x}_1)$$

and the following (finite) piece of information:

$$(\bar{x}_0 \restriction K, \bar{x}_1 \restriction K)$$

where $K = \{\sigma_0, \ldots, \sigma_{N(\xi)}\} \times N(\xi)$.

An application: maximal discrete sets

Let $\mathcal{R} \subseteq X^2$ (i.e. a binary relation on some set X).

Definition

We say a set $A \subseteq X$ is $\mathcal{R}$**-discrete** $\iff$

$$(\forall x, y \in A)\ x \neq y \Rightarrow \neg(x \,\mathcal{R}\, y).$$

Definition

We call such a set **maximal discrete** if it is not a proper subset of any discrete set.

$\mathcal{R}$ is maximal discrete *iff* $(\forall x \in X)(\exists a \in A)\ (x \,\mathcal{R}\, a) \vee (a \,\mathcal{R}\, a)$.

Example: Orthogonality of measures

- Let X be a standard Borel space.
- Consider $P(X)$, the standard Borel space of Borel probability measures on X.
- Two measures $\mu, \nu \in P(X)$ are said to be orthogonal, written

$$\mu \perp \nu$$

exactly if: there is a Borel set $A \subseteq X$ such that

$$\mu(A) = 1$$

and

$$\nu(A) = 0.$$

Maximal orthogonal families of measures

- We abbreviate **"maximal orthogonal family"** by **"mof"**.
- We restrict our attention to the case $X = 2^\omega$ from now on.
- Note that $P(2^\omega)$ is an effective Polish space.

History of maximal orthogonal families

Question (Mauldin, circa 1980)

Can a **mof** in $P(2^\omega)$ be analytic?

The answer turned out to be 'no':

Theorem (Preiss-Rataj, 1985)

There is no analytic ***mof*** *in* $P(2^\omega)$*.*

This is optimal, in a sense:

Theorem (Fischer-Törnqust, 2009)

In $\mathbf{L}$*, there is a* Π^1_1 ***mof*** *in* $P(2^\omega)$*.*

Mofs and forcing

Mofs are fragile creatures:

Facts

1. Adding any real destroys maximality of **mofs** from the groundmodel (observed by Ben Miller; not restricted to forcing extensions)
2. If there is a Cohen real over **L**, there are no Σ^1_2 **mofs** in $P(2^\omega)$ (F-T, 2009)
3. The same holds if there is a random real over **L** (Fischer-Friedman-Törnquist, 2010).
4. The same holds if there is a Mathias real over **L** (S-Törnquist, 2014).

Question (F-T, 2009)

If there is a Π^1_1 **mof**, does it follow that $\mathcal{P}(\omega) \subseteq \mathbf{L}$?

Π^1_1 mofs in extensions of $\mathbf{L}$

Theorem (S-Törnquist, 2014)

*If s is Sacks over $\mathbf{L}$ there is a (lightface!) Π^1_1 **mof** in $\mathbf{L}[s]$.*

Theorem (S 2015)

*The statement 'there is a Π^1_1 **mof** ' is consistent with $2^\omega = \omega_2$.*

In fact :

Theorem (S 2015)

Let $\mathcal{R}$ be a Σ^1_1 relation on an effective Polish space X. If $\bar{s}$ is generic for iterated Sacks forcing over $\mathbf{L}$, there is a (lightface) Δ^1_2 maximal $\mathcal{R}$-discrete set in $\mathbf{L}[\bar{s}]$.

Thank You!