Locally compact normal spaces:
ω_1-compact vs. σ-countably compact

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A *space of countable extent*, also called an ω_1-*compact space*, is one in which every closed discrete subspace is countable.

Here is one interesting unsolved problem about these concepts:

Problem 1. Is there a ZFC example of a locally compact, ω_1-compact space of cardinality $\aleph_1$ that is not σ-countably compact? one that is normal?

More generally, there is the question of what is the minimum cardinality of such spaces in ZFC. Examples of cardinality $\mathfrak{c}$ were obtained in 1975 by Erik van Douwen [vD] but the following improvement seems to be new:

Theorem 1. [Ny2] *There is a locally compact, normal, ω_1-compact space of cardinality $\mathfrak{b}$ that is not σ-countably compact.*

Higher separation axioms tell a different story.

Theorem 2. [Ny1] *In MM(S)[S] models, every locally compact, hereditarily normal,* ω_1*-compact space is* σ*-countably compact,* i.e., *the union of countably many countably compact subspaces.*

In stark contrast, we also have:

Theorem 3. [Ny1] *If* $\clubsuit$*, then there exists a locally compact, locally countable (hence first countable, and scattered)* ω_1*-compact, monotonically normal space of cardinality* $\aleph_1$ *that is not* σ*-countably compact.*

MM(S)[S] models require large cardinal axioms, whereas $\clubsuit$ does not, and monotonically normal spaces are hereditarily collectionwise normal and hereditarily countably paracompact. The questions suggested by the following problem are all unanswered:

Problem 2. Which of the numerous independence results implied by Theorems 2 and 3 requires large cardinal axioms?

Corollary 1. *Perhaps modulo large cardinals, it is ZFC-independent whether every locally compact, monotonically normal,* ω_1*-compact space is* σ*-countably compact.*

Of course, one could add "locally countable" and "of cardinality $\aleph_1$" to the listed properties. In the opposite direction, we can substitute the much weaker "hereditarily normal" for "monotonically normal" in this corollary.

Corollary 1 is unusual in that most independence results on monotonically normal spaces depend on whether Souslin's Hypothesis (SH) is true, and do not depend on large cardinal axioms. Here, it is not known whether either SH or its negation affect either direction in this independence result, nor whether its reliance on large cardials can be dropped.

Definition 1. A space X is *monotonically normal* provided that there is an operator $G(\_,\_)$ assigning to each ordered pair $\langle F_0, F_1\rangle$ of disjoint closed subsets an open set $G(F_0, F_1)$ such that
(a) $F_0 \subset G(F_0, F_1)$
(b) If $F_0 \subset F_0'$ and $F_1' \subset F_1$ then $G(F_0, F_1) \subset G(F_0', F_1')$
(c) $G(F_0, F_1) \cap G(F_1, F_0) = \emptyset$.

Theorem 2 is one of three related results in the next theorem, which uses the following concepts:

Definition 2. Given a subset D of a set X, an *expansion* of D is a family $\{U_d : d \in D\}$ of subsets of X such that $U_d \cap D = d$ for all $d \in D$. A space X is *[strongly] collectionwise Hausdorff* (abbreviated *[s]cwH*) if every closed discrete subspace has an expansion to a disjoint [*resp.* discrete] collection of open sets.

The properties of ω_1-*[s]cwH* only require taking care of those D that are of cardinality $\leq \omega_1$.

A well-known, almost trivial fact is that every normal, [ω_1]-cwH space is [ω_1]-scwH.

The following generalization of Theorem 2 has a slight abuse of language with the expressions PFA(S)[S] and MM(S)[S] as though they were axioms.

Theorem 3$^+$. *Let X be a locally compact, ω_1-compact space. If either*

(1) *X is monotonically normal and the LCT axiom holds, or*

(2) *X is hereditarily ω_1-scwH, in a model of either PFA or PFA(S)[S] or MM(S)[S], or*

(3) *X is hereditarily normal, in a model of MM(S)[S],*

then X is σ-countably compact, and is either Lindelöf or contains a copy of ω_1.

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The LCT axiom is a purely topological axiom:

The Locally Compact Trichotomy (LCT) axiom.

Every locally compact space has either:

(i) A countable collection of ω-bounded subspaces whose union is the whole space OR

(ii) An uncountable closed discrete subspace or

(iii) A countable subset with non-Lindelöf closure.

An *ω-bounded* space is one in which every countable subset has compact closure, and clearly such a space is countably compact.

The first conclusion for (1) follows from the fact that, in a monotonically normal space, every countable subset has Lindelöf closure [O]. This is also true of hereditarily ω_1-scwH spaces under MA(ω_1) and MA(S)[S], and hence PFA, PFA(S)[S] and MM(S)[S]. Only the last requires large cardinal axioms.

It is only for the consistency of the LCT axiom and for Theorem 2 — (3) in Theorem 2^+ — that large cardinals seem to be unavoidable. The LCT axiom is shown in [EN] to be a consequence of the Prime Ideal Dichotomy (PID) axiom, which uses the following concept:

Definition 3. *A P-ideal of countable sets* is a family $\mathcal{P}$ of sets such that, for every countable subfamily $\mathcal{Q}$ of $\mathcal{P}$, there exists $P \in \mathcal{P}$ such that $Q \subset^* P$ for every $Q \in \mathcal{Q}$. Here $Q \subset^* P$ means that $Q \setminus P$ is finite.

The PID states that, for every P-ideal $\mathcal{I}$ of subsets of a set X, either

(1) there is an uncountable $A \subset X$ such that every countable subset of A is in $\mathcal{I}$, or

(2) X is the union of countably many sets $\{B_n : n \in \omega\}$ such that $B_n \cap I$ is finite for all n and all $I \in \mathcal{I}$.

The PID was shown to imply the LCA in [EN]. In particular, *(i)* goes with (2), *(ii)* goes with (1), and *(iii)* goes with the collection of all countable closed discrete subspaces failing to form a P-ideal.

LCT: *Every locally compact space has either:*

(i) A countable collection of ω-bounded subspaces whose union is the whole space OR

(ii) An uncountable closed discrete subspace or

(iii) A countable subset with non-Lindelöf closure.

Theorem 3. *Let X be a locally compact, ω_1-compact, normal, hereditarily ω_1-scwH space. If either PFA or PFA(S)[S] holds, then X is countably paracompact.*

Proof. A normal space X is countably paracompact if, and only if, for each descending sequence of closed sets $\langle F_n \rangle_{n=1}^{\infty}$ with empty intersection, there is a sequence of open sets $\langle U_n \rangle_{n=1}^{\infty}$ with empty intersection, with $F_n \subset U_n$ for all n [W, 21.3]. If X is a countable union of countably compact subsets C_m, then in such a sequence of closed sets F_n, we can only have $F_n \cap C_m \neq \emptyset$ for finitely many n. [Otherwise, countable compactness of C_m implies $\bigcap_{n=1}^{\infty} C_m \cap F_n \neq \emptyset$.] In any Tychonoff space, every pseudocompact subspace, and hence every countably compact subspace, has pseudocompact closure, and every normal, pseudocompact space is countably compact [W, 17J 3.]; and so the complements of the sets $\overline{C_m}$ form the desired sequence of open sets. $\square$

The equivalence in the preceding proof is due to Dowker, who also showed its equivalence with $X \times [0, 1]$ being normal. In honor of his pioneering work, normal spaces that are *not* countably paracompact are called "Dowker spaces." This results in a bit of economy of wording. For instance, Theorem 3 gives the consistency of the statement that every locally compact hereditarily ω_1-scwH Dowker space has an uncountable discrete subspace. This makes a nice companion to the following recent theorem of Dow and Tall:

Theorem. *If MM(S)[S], then every locally compact Dowker space of Lindelöf number* $\leq A_1$ *includes a perfect preimage of* ω_1.