The Kechris-Pestov-Todorčević correspondence in an abstract setting

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Starting point

A. S. KECHRIS, V. G. PESTOV, S. TODORČEVIĆ: *Fraïssé limits, Ramsey theory and topological dynamics of automorphism groups.* GAFA Geometric and Functional Analysis, 15 (2005) 106–189.

- Fraïssé Theory
- Structural Ramsey Theory
- Topological Dynamics of Aut gp's

Starting point

A. S. KECHRIS, V. G. PESTOV, S. TODORČEVIĆ: *Fraïssé limits, Ramsey theory and topological dynamics of automorphism groups.* GAFA Geometric and Functional Analysis, 15 (2005) 106–189.

Thesis. *Category theory is an appropriate context for implementing the Kechris-Pestov-Todorčević correspondence.*

Outline

Homogeneity	•	Ramsey prop	•	Extreme amenability

↓

↓ *abstract interpretation*

↓

Homogeneity in a category	•	Ramsey prop in a category	•	Extreme amenability w.r.t. particular topology

Outline

Homogeneity •	Ramsey prop •	Extreme amenability

↓
↓ *abstract interpretation*
↓

Homogeneity in a category •	Ramsey prop in a category •	Extreme amenability w.r.t. particular topology

↓
↓ *specialization*
↓

Homogeneity • Ramsey prop • Extreme amenability
for ultrahomog structs that are not Fraïssé limits
(e.g. uncountable ulrahomog structs)

Outline

Homogeneity •	Ramsey prop •	Extreme amenability

↓
↓ *abstract interpretation*
↓

Homogeneity in a category •	Ramsey prop in a category •	Extreme amenability w.r.t. particular topology

↓
↓ *categorical duality*
↓

Projective Homogeneity •	Dual Ramsey prop •	Extreme amenability

T. IRWIN, S. SOLECKI: *Projective Fraïssé limits and the pseudo-arc.* Trans. Amer. Math. Soc. 358, no. 7 (2006) 3077–3096.

Ramsey property in a category

For $k \geqslant 2$ and $A, B, C \in \mathrm{Ob}(\mathbb{C})$ write $C \longrightarrow (B)^A_k$ if:

- $\hom(A, B) \neq \varnothing$ and $\hom(B, C) \neq \varnothing$, and
- for every **Set**-mapping $\chi : \hom(A, C) \to k$ there is a $\mathbb{C}$-morphism $w : B \to C$ such that $|\chi(w \cdot \hom(A, B))| = 1$.

A category $\mathbb{C}$ has the Ramsey property if:

for all $k \geqslant 2$ and all $A, B \in \mathrm{Ob}(\mathbb{C})$ such that $\hom(A, B) \neq \varnothing$ there is a $C \in \mathrm{Ob}(\mathbb{C})$ such that $C \longrightarrow (B)^A_k$.

Ramsey property in a category

A category $\mathbb{C}$ has the dual Ramsey property if $\mathbb{C}^{\mathrm{op}}$ has the Ramsey property.

Recall. The oposite category $\mathbb{C}^{\mathrm{op}}$:

1. objects of $\mathbb{C}^{\mathrm{op}}$ are the objects of $\mathbb{C}$;
2. $\hom_{\mathbb{C}^{\mathrm{op}}}(A, B) = \hom_{\mathbb{C}}(B, A)$;
3. $\underset{\text{in } \mathbb{C}^{\mathrm{op}}}{f \cdot g} = \underset{\text{in } \mathbb{C}}{g \cdot f}$

$$(A \xleftarrow{g} B) \cdot (B \xleftarrow{f} C) = A \xleftarrow{f \cdot g} C$$

Ramsey property and extremely amenable groups

A. S. KECHRIS, V. G. PESTOV, S. TODORČEVIĆ: *Fraïssé limits, Ramsey theory and topological dynamics of automorphism groups.* GAFA Geometric and Functional Analysis, 15 (2005) 106–189.

Theorem. *TFAE for a countable locally finite ultrahomogeneous first-order structure F:*

1. Aut(F) *is extremely amenable*
2. Age(F) *has the Ramsey property and consists of rigid elements.*

- A group G is *extremely amenable* if every continuous action of G on a compact Hausdorff space X has a common fixed point.

KPT theory in a category – the setup

Let $\mathbb{C}$ be a category and $\mathbb{C}_0$ a full subcategory of $\mathbb{C}$ such that:

(C1) all morphisms in $\mathbb{C}$ are monic ($=$ left cancellable);

(C2) $\mathrm{Ob}(\mathbb{C}_0)$ is a set;

(C3) for all $A, B \in \mathrm{Ob}(\mathbb{C}_0)$ the set $\hom(A, B)$ is finite;

(C4) for every $F \in \mathrm{Ob}(\mathbb{C})$ there is an $A \in \mathrm{Ob}(\mathbb{C}_0)$ such that $A \to F$;

(C5) for every $B \in \mathrm{Ob}(\mathbb{C}_0)$ the set $\{A \in \mathrm{Ob}(\mathbb{C}_0) : A \to B\}$ is finite.

$\mathbb{C}_0$ are (templates of) *finite objects* in $\mathbb{C}$.

$\mathrm{Age}(F) = \{A \in \mathrm{Ob}(\mathbb{C}_0) : A \to F\}$.

KPT theory in a category – the setup

Example. $\mathbf{Rel}(\Delta)$

- objects are all relational structures of type Δ,
- $\hom(A, B) =$ embeddings $A \to B$,
- $\mathbf{Rel}(\Delta)_0$ objects are finite relational structures $R = (\{1, \ldots, n\}, \Delta^R)$, $n \geqslant 1$.

KPT theory in a category – the setup

Example. Haus

- objects are Hausdorff spaces,
- $\hom(A, B) =$ continuous surjective maps $A \to B$,
- $\mathbf{Haus}_0$ objects are finite discrete spaces $\{1, \ldots, n\}$, $n \geqslant 1$.

An age of a structure in an op-category will be referred to as the projective age and denoted by $\partial\mathrm{Age}(A)$.

Example. $\mathcal{K} =$ Cantor set 2^ω.
$\partial\mathrm{Age}(\mathcal{K}) =$ all finite discrete spaces in $\mathbf{Haus}^{\mathrm{op}}$.

KPT theory in a category – the setup

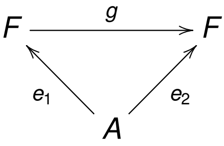
Example. OHaus

- objects are all lin ordered Hausdorff spaces,
- $\hom(A, B) =$ continuous monotonous surjective maps $A \to B$,
- $\mathbf{OHaus}_0$ objects are finite chains $(\{1, \dots, n\}, \leqslant)$, $n \geqslant 1$.

Example. $\mathcal{K}_{\leqslant} = \mathcal{K}$ with the lexicographic order.
$\partial\mathrm{Age}(\mathcal{K}_{\leqslant}) =$ all finite chains in $\mathbf{OHaus}^{\mathrm{op}}$.

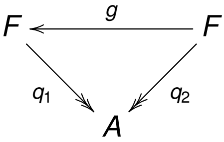
Homogeneous objects

$F \in \mathrm{Ob}(\mathbb{C})$ is homogeneous if for every $A \in \mathrm{Age}(F)$ and every pair of morphisms $e_1, e_2 : A \to F$ there is a $g \in \mathrm{Aut}(F)$ such that $g \cdot e_1 = e_2$.



Example. Ultrahomogeneous structures in “direct” categories.

Following Irwin and Solecki, homogeneous structures in an op-category will be referred to as projectively homogeneous.

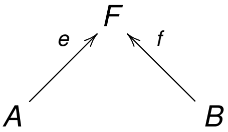


Example. Both $\mathcal{K}$ and $\mathcal{K}_{\leqslant}$ are projectively homogeneous (each in its category).

Locally finite objects

$F \in \mathrm{Ob}(\mathbb{C})$ is locally finite if

1. for every $A, B \in \mathrm{Age}(F)$ and every $e : A \to F$, $f : B \to F$ there are a $D \in \mathrm{Age}(F)$, $r : D \to F$, $p : A \to D$ and $q : B \to D$ such that $r \cdot p = e$ and $r \cdot q = f$, and
2. for every $H \in \mathrm{Ob}(\mathbb{C})$, $r' : H \to F$, $p' : A \to H$ and $q' : B \to H$ such that $r' \cdot p' = e$ and $r' \cdot q' = f$ there is an $s : D \to H$ such that the diagram below commutes.



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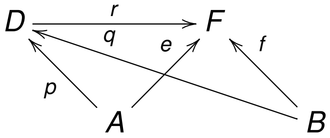

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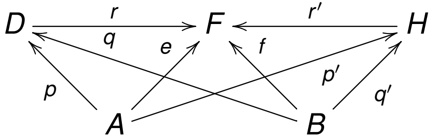

Locally finite objects

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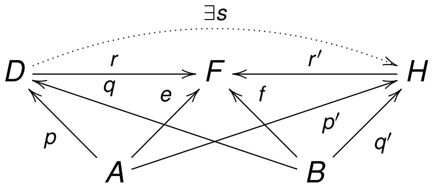

Locally finite objects

Example. Every object in **Rel**(Δ) is locally finite.

Locally finite structures in an op-category will be referred to as projectively locally finite.

Example. Both $\mathcal{K}$ and $\mathcal{K}_{\leqslant}$ are projectively locally finite (each in its category).

Finitely separated automorphisms

The automorphisms of $F \in \mathrm{Ob}(\mathbb{C})$ are finitely separated if the following holds for all $f, g \in \mathrm{Aut}(F)$:

> if $f \neq g$ then there is an $A \in \mathrm{Age}(F)$ and an $e : A \to F$ such that $f \cdot e \neq g \cdot e$.

Example. Automorphisms of every relational structure are finitely separated.

Example. The automorphisms of both $\mathcal{K}$ and $\mathcal{K}_{\leqslant}$ are finitely separated (each in its category).

The topology generated by the age of an object

$F \in \mathrm{Ob}(\mathbb{C})$

For $A \in \mathrm{Age}(F)$ and $e_1, e_2 \in \hom(A, F)$ let

$$N_F(e_1, e_2) = \{f \in \mathrm{Aut}(F) : f \cdot e_1 = e_2\}.$$

Lemma. *Let F be a locally finite object in $\mathbb{C}$. Then*

$$\mathcal{M}_F = \{N_F(e_1, e_2) : A \in \mathrm{Age}(F); e_1, e_2 \in \hom(A, F)\}$$

is a base of a topology α_F on $\mathrm{Aut}(F)$. If, in addition, the automorphisms of F are fintely separated, $\mathrm{Aut}(F)$ endowed with the topology α_F is a Hausdorff topological group.

The topology generated by the age of an object

Example. In **Rel**(Δ): α_A is the pointwise convergence topology for every Δ-structure A.

Example. In **Haus**^{op}: $\alpha_{\mathcal{K}} =$ compact-open topology on $\mathcal{K}$.

Example. In **OHaus**^{op}: $\alpha_{\mathcal{K}_\leqslant} =$ "compact interval-open interval" topology on $\mathcal{K}_\leqslant$.

Ramsey property and extreme amenability

Theorem. *Let F be a homogeneous locally finite object in $\mathbb{C}$ whose automorphisms are finitely separated. TFAE:*

1. Aut(F) *endowed with α_F is extr amenable,*
2. Age(F) *has the Ramsey property.*

Ramsey property and extreme amenability

Theorem. *Let F be a homogeneous locally finite object in $\mathbb{C}$ whose automorphisms are finitely separated. TFAE:*

1. $\mathrm{Aut}(F)$ *endowed with* α_F *is extr amenable,*
2. $\mathrm{Age}(F)$ *has the Ramsey property.*

Corollary 1. *Let F be an ultrahomogeneous relational structure. Then* $\mathrm{Aut}(F)$ *with with the pointwise convergence topology is extremely amenable if and only if* $\mathrm{Age}(F)$ *has the Ramsey property.*

D. Bartošová: *Universal minimal flows of groups of automorphisms of uncountable structures.* Canadian Mathematical Bulletin, 2012.

Ramsey property and extreme amenability

Theorem. *Let F be a homogeneous locally finite object in $\mathbb{C}$ whose automorphisms are finitely separated. TFAE:*

1. $\mathrm{Aut}(F)$ *endowed with* α_F *is extr amenable,*
2. $\mathrm{Age}(F)$ *has the Ramsey property.*

Example. The automorphism group of every ultrahomogeneous chain, endowed with the pointwise convergence topology, is extremely amenable.

For $(\mathbb{Q}, \leqslant)$: V. G. PESTOV: *On free actions, minimal flows and a problem by Ellis.* Transactions of the American Mathematical Society, 350 (1998) 4149–4165.

In general for chains: D. BARTOŠOVÁ: *Universal minimal flows of groups of automorphisms of uncountable structures.* Canadian Mathematical Bulletin, 2012.

Ramsey property and extreme amenability

Theorem. *Let F be a homogeneous locally finite object in $\mathbb{C}$ whose automorphisms are finitely separated. TFAE:*

1. $\mathrm{Aut}(F)$ *endowed with* α_F *is extr amenable,*
2. $\mathrm{Age}(F)$ *has the Ramsey property.*

Corollary 2. *Let F be a projectively locally finite projectively homogeneous structure. Then* $\mathrm{Aut}(F)$ *endowed with the topology* α_F *is extremely amenable if and only if* $\partial\mathrm{Age}(F)$ *has the dual Ramsey property.*

Ramsey property and extreme amenability

Theorem. *Let F be a homogeneous locally finite object in* $\mathbb{C}$ *whose automorphisms are finitely separated. TFAE:*

1. Aut(F) *endowed with* α_F *is extr amenable,*
2. Age(F) *has the Ramsey property.*

Corollary 3. *Let F be a projectively homogeneous 0-dimensional Hausdorff space. Then* Homeo(F) *endowed with the compact-open topology is extremely amenable if and only if* ∂Age(F) *has the dual Ramsey property.*

(Cf. D. BARTOŠOVÁ: *Universal minimal flows of groups of automorphisms of uncountable structures.* Canadian Mathematical Bulletin, 2012.)

Ramsey property and extreme amenability

Theorem. *Let F be a homogeneous locally finite object in $\mathbb{C}$ whose automorphisms are finitely separated. TFAE:*

1. Aut(F) *endowed with* α_F *is extr amenable,*
2. Age(F) *has the Ramsey property.*

Example. In **Haus**$^{\text{op}}$: Homeo($\mathcal{K}$) endowed with the compact-open topology is not extremely amenable.

Example. In **OHaus**$^{\text{op}}$: Let G be the homeomorphism group of $\mathcal{K}_{\leqslant}$ endowed with $\alpha_{\mathcal{K}_{\leqslant}} =$ "*compact interval – open interval*" topology. Then G is extremely amenable.

Minimal flows and the expansion property

A. S. KECHRIS, V. G. PESTOV, S. TODORČEVIĆ: *Fraïssé limits, Ramsey theory and topological dynamics of automorphism groups.* GAFA Geometric and Functional Analysis, 15 (2005) 106–189.

Theorem. *Let $\mathcal{F}$ be a locally finite Fraïssé structure, $\mathcal{F}^*$ a Fraïssé order expansion of $\mathcal{F}$ and X^* the set of admissible linear orders on F. TFAE:*

1. *X^* is a minimal* $\mathrm{Aut}(\mathcal{F})$*-flow*
2. $\mathrm{Age}(\mathcal{F}^*)$ *has the ordering property w.r.t.* $\mathrm{Age}(\mathcal{F})$*.*

Minimal flows and the expansion property

L. NGUYEN VAN THÉ: *More on the Kechris-Pestov-Todorcevic correspondence: precompact expansions.* Fund. Math. 222 (2013), 19–47.

Theorem. *Let $\mathcal{F}$ be a locally finite Fraïssé structure, $\mathcal{F}^*$ a Fraïssé precompact expansion of $\mathcal{F}$ and X^* the set of admissible expansions on F. TFAE:*

1. *X^* is a minimal* $\mathrm{Aut}(\mathcal{F})$*-flow*
2. $\mathrm{Age}(\mathcal{F}^*)$ *has the expansion property w.r.t.* $\mathrm{Age}(\mathcal{F})$*.*

Minimal flows and the expansion property

$\Theta = (\theta_i)_{i<n}$ – a **finite** relational language

$\Omega_F = \bigcup\{\mathrm{hom}(A, F) : A \in \mathrm{Ob}(\mathbb{C}_0)\}$

For $F \in \mathrm{Ob}(\mathbb{C})$, a Θ-expansion of F is a tuple $(F, (\rho_i)_{i<n})$ where ρ_i is a finitary relation on Ω_F.

Lemma. *Ω_A is finite for $A \in \mathrm{Ob}(\mathbb{C}_0)$.*

So, Θ-finite $\Longrightarrow$ these expansions are always precompact.

Minimal flows and the expansion property

$\mathbb{C}(\Theta)$ – a category of Θ expansions of objects from $\mathbb{C}$:

- objects are Θ-expansions of objects from $\mathbb{C}$;
- $f : (F, (\rho_i)_{i<n}) \to (H, (\sigma_i)_{i<n})$ is a $\mathbb{C}(\Theta)$-morphism if
  - $f \in \hom_{\mathbb{C}}(F, H)$, and
  - $(e_0, \ldots, e_{m-1}) \in \rho_i \Rightarrow (f \cdot e_0, \ldots, f \cdot e_{m-1}) \in \sigma_i$, for all $i < n$.

$\mathrm{Age}(F, (\theta_i)_{i<n})$ has the expansion property w.r.t. $\mathrm{Age}(F)$ if

for every $A \in \mathrm{Age}(F)$ there is a $B \in \mathrm{Age}(F)$ such that for all $(A, (\rho_i)_{i<n}), (B, (\sigma_i)_{i<n}) \in \mathrm{Age}(F, (\theta_i)_{i<n})$ we have a morphism $(A, (\rho_i)_{i<n}) \to (B, (\sigma_i)_{i<n})$ in $\mathbb{C}(\Theta)$.

Minimal flows and the expansion property

$F \in \mathrm{Ob}(\mathbb{C})$, $G = \mathrm{Aut}(F)$

$E_F = \{$all the tuples $(\rho_i)_{i<n}$ where $\rho_i \subseteq \Omega_{\mathcal{F}}^{m_i}\}$

G acts on E_F logically, that is

$$(\rho_i)_{i<n}^g = (\rho_i^g)_{i<n} \quad \text{and}$$
$$(e_0, \ldots, e_{m-1}) \in \rho_i^g \Rightarrow (g^{-1} \cdot e_0, \ldots, g^{-1} \cdot e_{m-1}) \in \rho_i$$

Minimal flows and the expansion property

Theorem. *Let F be a locally finite homogeneous object in $\mathbb{C}$ and let $G = \mathrm{Aut}(F)$. Let $(F, (\rho_i)_{i<n})$ be a Θ-expansion of F which is locally finite in $\mathbb{C}(\Theta)$. Let $X^\Theta = \overline{(\rho_i)^G_{i<n}}$ be a G-flow where the action of G is logical. TFAE:*

1. *X^Θ is a minimal G-flow.*
2. *$\mathrm{Age}(F, (\rho_i)_{i<n})$ has the expansion property w.r.t. $\mathrm{Age}(F)$.*

Example. Let S be an infinite set, let $G = \mathrm{Sym}(S)$ and let $(S, \leqslant)$ be an ultrahomogeneous chain. Then

$$X^\Theta = \overline{\leqslant^G} = \text{all lin orders on } S$$

is a minimal G-flow.

Minimal flows and the expansion property

Theorem. *Let F be a locally finite homogeneous object in $\mathbb{C}$ and let $G = \text{Aut}(F)$. Let $(F, (\rho_i)_{i<n})$ be a Θ-expansion of F which is locally finite in $\mathbb{C}(\Theta)$. Let $X^\Theta = \overline{(\rho_i)^G_{i<n}}$ be a G-flow where the action of G is logical. TFAE:*

1. *X^Θ is a minimal G-flow.*
2. $\text{Age}(F, (\rho_i)_{i<n})$ *has the expansion property w.r.t.* $\text{Age}(F)$.

Corollary. *Let F be a projectively locally finite projectively homogeneous object and let $G = \text{Aut}(F)$. Let $(F, (\rho_i)_{i<n})$ be a Θ-expansion of F which is projectively locally finite. Let $X^\Theta = \overline{(\rho_i)^G_{i<n}}$ be a G-flow where the action of G is logical. TFAE:*

1. *X^Θ is a minimal G-flow.*
2. $\partial\text{Age}(F, (\rho_i)_{i<n})$ *has the exp prop w.r.t.* $\partial\text{Age}(F)$.

Universal minimal flows

A. S. KECHRIS, V. G. PESTOV, S. TODORČEVIĆ: *Fraïssé limits, Ramsey theory and topological dynamics of automorphism groups.* GAFA Geometric and Functional Analysis, 15 (2005) 106–189.

Theorem. *Let $\mathcal{F}$ be a locally finite Fraïssé structure, $\mathcal{F}^*$ a Fraïssé order expansion of $\mathcal{F}$ and X^* the set of admissible linear orders on F. TFAE:*

1. *X^* is the universal minimal* $\mathrm{Aut}(\mathcal{F})$*-flow*
2. $\mathrm{Age}(\mathcal{F}^*)$ *has the Ramsey property and the ordering property w.r.t.* $\mathrm{Age}(\mathcal{F})$*.*

Universal minimal flows

Theorem. *Let F be a locally finite homogeneous object in $\mathbb{C}$ and let $G = \text{Aut}(F)$. Let $(F, (\rho_i)_{i<n})$ be a Θ-expansion of F which is locally finite and homogeneous in $\mathbb{C}(\Theta)$. Let $X^\Theta = \overline{(\rho_i)^G_{i<n}}$ be a G-flow where the action of G is logical.*

If X^Θ is the universal minimal G-flow then $\text{Age}(F, (\rho_i)_{i<n})$ has the Ramsey property and the expansion property w.r.t. $\text{Age}(F)$.