A fragment of PFA consistent with large continuum

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A partition relation

Definition: $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$

For every graph $[\omega_1]^2 = K_0 \cup K_1$, one of the two happens:

(i) there is an uncountable $A \subseteq \omega_1$ such that $[A]^2 \subseteq K_0$, or

(ii) there is an uncountable $A \subseteq \omega_1$ and an uncountable pairwise disjoint $\mathcal{B} \subseteq [\omega_1]^{<\omega}$ such that for every $\alpha \in A$ and $F \in \mathcal{B}$, if $\alpha < F$, then $\{\alpha\} \otimes F \cap K_1 \neq \emptyset$.

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Theorem (Todorčević)

(i) PFA implies $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$;

(ii) $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$ implies that there are no S-spaces;

(iii) ($\mathfrak{p} > \omega_1$ or $\mathfrak{b} > \omega_1$) $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$ implies $\omega_1 \to (\omega_1, \alpha)^2$ for every ordinal $\alpha < \omega_1$.

The side condition method

The poset: $p = (w_p, \mathcal{N}_p)$ where

(i) $w_p \subseteq \omega_1$ a finite 0-clique;

(ii) $\mathcal{N}_p$ a finite $\in$-chain of elementary substructures of $(H_\theta, \in, \triangleleft)$ containing K_0, K_1;

(iii) For every $\{\alpha, \beta\}_<$ in w_p, there is $M \in \mathcal{N}_p$ such that $M \cap \{\alpha, \beta\}_< = \{\alpha\}$.

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Proving properness

For $n < \omega$, let $\mathscr{H}^n$ be the n-fold Fubini product of the coideal of uncountable subsets of ω_1. If $\mathcal{F} \in \mathscr{H}^n \cap M_0 \in M_1 \in M_2 \ldots \in M_n$, $\bar{v} \in \mathcal{F}$ is separated by $M_0 \in M_1 \in M_2 \ldots \in M_n$, then there is $\bar{u} \in M_0$ such that $\bar{u} \cup \bar{v}$ is a 0-clique.

An axiom

Definition: GID_{ω_1}

Let $[\omega_1]^2 = K_0 \cup K_1$ be a graph on ω_1. Let $\mathscr{I}$ be a proper ideal on ω_1 which is σ-generated by $\langle I_\alpha \colon \alpha < \omega_1 \rangle$ such that for any $n < \omega$, if $\mathcal{F} \in \mathscr{H}^n$, then there are $\bar{u} < \bar{v}$ in $\mathcal{F}$ such that $\bar{u} \otimes \bar{v} \subseteq K_0$. Then there is an uncountable 0-clique.

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PFA implies GID_{ω_1}: $p = (w_p, \mathcal{N}_p)$ where

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Application

(GID_{ω_1}) If X is a second countable T_2 space of size $\aleph_1$, $K \subseteq [X]^2$ an open graph such that the σ-ideal of countably chromatic sets is proper and $\aleph_1$-generated, then X has an uncountable clique.

Main result

Theorem (I.)

(CH) Let κ be a regular cardinal such that $2^{<\kappa} = \kappa$ and $\kappa^{\aleph_1} = \kappa$. Then there is an $\aleph_2$-Knaster, proper partial order $\mathbb{P}$ such that

$$V^{\mathbb{P}} \models \mathsf{GID}_{\omega_1} + \mathsf{MA}_{\omega_1} + 2^{\aleph_0} = \kappa.$$

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Method

Asperó-Mota iterations: Build an iteration $\langle \mathbb{P}_\alpha : \alpha \leq \kappa \rangle$ with symmetric systems of structures as side conditions. Also need symmetries at individual stages, and a sequence of increasingly correct truth predicates for definability of the forcing at individual stages.

Symmetric systems

Definition: Symmetric systems

Let $T \subseteq H(\kappa)$ and $\mathcal{N}$ a finite set of countable subsets of $H(\kappa)$. Then $\mathcal{N}$ is a T-*symmetric system* if the following hold:

(i) if $N \in \mathcal{N}$, then $(N, \in, T) \prec (H(\kappa), \in, T)$;

(ii) if $N, N' \in \mathcal{N}$ are such that $\delta_N = \delta_{N'}$ then there is a unique isomorphism

$$\Psi_{N,N'} : (N, \in, T) \to (N', \in, T)$$

which is the identity on $N \cap N'$;

(iii) if $N, N', M \in \mathcal{N}$ are such that $\delta_N = \delta_{N'}$ and $M \in N$, then $\Psi_{N,N'}(M) \in \mathcal{N}$;

(iv) if $M, N \in \mathcal{N}$ are such that $\delta_M < \delta_N$, then there is $N' \in \mathcal{N}$ such that $M \in N'$ and $\delta_N = \delta_{N'}$.

Why symmetric systems?

Lemma

Let $\mathcal{N}$ be a symmetric system and let $N \in \mathcal{N}$. Then there is $\mathcal{M} \subseteq \mathcal{N}$ a finite $\in$-chain such that

(i) N is the lowest model of $\mathcal{M}$;

(ii) If $M \in \mathcal{N}$ is such that $\delta_M > \delta_N$, then there is $M' \in \mathcal{M}$ such that $\delta_M = \delta_{M'}$.

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Lemma

Let $\mathscr{I}$ be an ideal on ω_1 which is σ-generated by $\langle I_\alpha : \alpha < \omega_1 \rangle$.

(i) If $M, N \prec (H_\theta, \in, \lhd)$ containing $\langle I_\alpha : \alpha < \omega_1 \rangle$ are such that $\delta_M = \delta_N$, then for any $\{\alpha, \beta\}_< \subseteq \omega_1$, M separates $\{\alpha, \beta\}_<$ iff N separates $\{\alpha, \beta\}_<$.

(ii) If $\mathcal{N}$ is a symmetric system all elements of which contain $\langle I_\alpha : \alpha < \omega_1 \rangle$, if $w \in [\omega_1]^{<\omega}$ is separated by $\mathcal{N}$ and $N \in \mathcal{N}$ is a low model of $\mathcal{N}$, then there is $\mathcal{M} \subseteq \mathcal{N}$ as above which separates w.

Asperó-Mota iterations

Setup

Let $\Phi : \kappa \to H(\kappa)$ a nice surjection. Let $\langle \theta_\alpha : \alpha \leq \kappa \rangle$ be a fast growing sequence of regular cardinals such that $\theta_0 = |H(\beth_2(\kappa))|^+$, and for $\alpha \leq \kappa$, let

$$\mathcal{M}^*_\alpha = \{N^* \in [H(\theta_\alpha)]^{\aleph_0} : N^* \prec H(\theta_\alpha), \Phi, \lhd, \langle \theta_\beta : \beta < \alpha \rangle \in N^*\},$$

and $\mathcal{M}_\alpha = \{N^* \cap H(\kappa) : N^* \in \mathcal{M}^*_\alpha\}$. Let T^α be the $\lhd$-least $T \subseteq H(\kappa)$ such that for every $N \in [H(\kappa)]^{\aleph_0}$, if $(N, \in T) \prec (H(\kappa), \in, T)$, then $N \in \mathcal{M}_\alpha$. Let

$$\mathcal{T}_\alpha = \{N \in [H(\kappa)]^{\aleph_0} : (N, \in, T^\alpha) \prec (H(\kappa), \in, T^\alpha)\}.$$

Asperó-Mota iterations

The partial order

The poset is $\mathbb{P} = \mathbb{P}_\kappa$ where $\langle \mathbb{P}_\alpha \colon \alpha \leq \kappa \rangle$ is obtained in the following way. Let $\beta \leq \kappa$. Suppose that $\mathbb{P}_\alpha$ has been defined for every $\alpha < \beta$. Elements of $\mathbb{P}_\beta$ will be a pair $q = (F_q, \Delta_q)$ where

(C1) F_q is a finite function such that $\mathrm{dom}(F_q) \subseteq \beta$;

(C2) Δ_q is a finite set of pairs (N, γ) such that $N \in [H(\kappa)]^{\aleph_0}$, γ an ordinal such that $\gamma \leq \min\{\beta, \sup(N \cap \kappa)\}$;

(C3) $\mathcal{N}^q_\beta = \{N \colon (N, \beta) \in \Delta_q, \beta \in N\}$ is a T^β-symmetric system;

(C4) if $\alpha < \beta$, then $q|_\alpha = (F_q \restriction \alpha, \{(N, \min\{\alpha, \gamma\}) \colon (N, \gamma) \in \Delta_q\}) \in \mathbb{P}_\alpha$;

(C5) if $\xi \in \mathrm{dom}(F_q)$, then $F_q(\xi) \in H(\kappa)$ and $q|_\xi \Vdash$ "$F_q(\xi) \in \Upsilon(\xi)$";

(C6) if $\xi \in \mathrm{dom}(F_q), N \in \mathcal{T}_{\xi+1}$ and $(N, \nu) \in \Delta_q$ for some $\nu \geq \xi + 1$, then $q|_\xi \Vdash$ "$F_q(\xi)$ is $(N[G_\xi], \Upsilon(\xi))$-generic".

The iterands

The bookkeeping

Suppose we have defined $\mathbb{P}_\alpha$.

(i) If $\Phi(\alpha)$ is a $\mathbb{P}_\alpha$-name for a ccc poset, then $\Upsilon(\alpha) = \Phi(\alpha)$.

(ii) If $\Phi(\alpha)$ is a $\mathbb{P}_\alpha$-name for a graph $[\omega_1]^2 = K_0 \cup K_1$ and a proper ideal $\mathscr{I}$ which is σ-generated by $\langle I_\zeta : \zeta < \omega_1 \rangle$ which has no bad sets, then $\Upsilon(\alpha) = \mathbb{Q}^{K_0, K_1, \langle I_\zeta : \zeta < \omega_1 \rangle, V, G_\alpha}$.

(iii) Otherwise, $\Upsilon(\alpha)$ is the trivial poset.

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The iterand: $q \in \mathbb{Q}^{K_0, K_1, \langle I_\zeta : \zeta < \omega_1 \rangle, V, G_\alpha}$ if $q = (w_q, \mathcal{N}_q)$ where

1. $w_q \subseteq \omega_1$ is a finite 0-clique;
2. $\mathcal{N}_q \subseteq [H(\kappa)^V]^{\aleph_0}$ is, in V, a $T^{\alpha+1}$-symmetric system which separates w_q: for every $\xi < \nu$ in w_q, there is $M \in \mathcal{N}_q$ such that $\bigcup (M[G_\alpha] \cap \mathscr{I}) \cap \{\xi, \nu\} = \{\xi\}$.
3. There is some $p \in G_\alpha$ such that $\mathcal{N}_q \subseteq \mathcal{N}_\alpha^p$.

Properness

Propitiousness

In $V[G_\alpha]$, $\mathbb{Q}$ is *propitious* for V, G_α if there is a club $D \subseteq [H(\kappa)^V]^{\aleph_0}$, $D \in V$, such that if

(a) $q \in \mathbb{Q}$,

(b) $\mathcal{N} \subseteq D$ is a $T^{\alpha+1}$-symmetric system,

(c) $q \in N[G_\alpha]$ for some $N \in \mathcal{N}$ of minimal height,

(d) there is a $p \in G_\alpha$ such that $\mathcal{N} \subseteq \mathcal{N}_\alpha^p$,

then there is a $q^* \leq q$ which is $(N[G_\alpha], \mathbb{Q})$-generic for each $N \in \mathcal{N}$.

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Propitiousness of the iterands

1. (Shelah, Mekler) ccc posets are propitious;
2. The other iterands: inductively, but essentially the same as in the two lemmas.

Questions

Are any of the following consistent?

(i) Classification of directed posets/transitive relations + $2^{\aleph_0} > \aleph_2$.

(ii) (Shelah+Zapletal) Every poset of uniform density $\aleph_1$ embeds $\mathbb{C}(\aleph_1)$ + $2^{\aleph_0} > \aleph_2$.

(iii) (Fremlin BU) $\mathsf{MA}_{\omega_1} + \omega_1 \not\to (\omega_1, \alpha)^2$ for some countable ordinal α.
   - (Todorčević) Necessarily, $\alpha > \omega^2$.
   - (Abraham+Todorčević) $\mathsf{MA}_{\omega_1} + \omega_1 \not\to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$ is consistent.

(iv) RPFA^2 + $2^{\aleph_0} > \aleph_2$.

The end