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Choiceless Ramsey Theory for Linear Orders

Joint work with Philipp Lücke and Philipp Schlicht

Ramsey-Type problems have been considered both in finite and infinite combinatorics. In infinite combinatorics most attention has been payed to partition relations between cardinals assuming the Axiom of Choice and almost all research dealt with ordinals (We think of cardinals as initial ordinals here). A notable exception is [7]. There the authors prove the following theorem.

Theorem. *Assume the Axiom of Choice. Then for all order-types* ψ *we have* $\psi \not\longrightarrow (4, \omega^* + \omega)^3$, $\psi \not\longrightarrow (4, \omega + \omega^*)^3$ *and* $\psi \not\longrightarrow (5, \omega^* + \omega \vee \omega + \omega^*)^3$.

Together with the folklore result (using AC) that $\psi \not\longrightarrow (\omega, \omega^*)^2$ this puts things into perspective. It is known that one can have very strong partition properties in models of ZF violating AC, consider for example Mathias's result that $\omega \longrightarrow (\omega)^{\omega}_2$ is consistent with ZF—cf. [6] or Martin's discovery that AD implies $\omega_1 \longrightarrow (\omega_1)^{\omega_1}$ which failed to be published by him (but cf. [2, 3, 4, 5]).

We focus on linear orders of the form $\langle {}^{\alpha}2, <_{lex} \rangle$ for ordinals α and prove positive and negative partition relations, an example of the former is the following theorem.

Theorem. *It is consistent with ZF that* $\langle {}^{\alpha}2, <_{lex} \rangle \longrightarrow (\langle {}^{\alpha}2, <_{lex} \rangle)^2$.

In contrast, here is an example of a negative partition relation.

Theorem. $\langle {}^{\alpha}2, <_{lex} \rangle \not\longrightarrow (6, \kappa^* + \kappa \vee 2 + \kappa^* \vee \kappa 2 \vee \omega\omega^*)^4$ *for all initial ordinals* κ *and all ordinals* $\alpha < \kappa^+$.

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