Robert Ratowski

Wrocław Univerity of Technology, Poland
robert.ralowski@pwr.edu.pl

Regular familes of small subsets of Polish spaces

Joint work with Szymon Żeberski

We consider a nice parametrised families of sets from fixed σ-ideal $\mathcal{I}$ defined on Polish space X. The covering number (i.e smallest size of family of sets from $\mathcal{I}$ for which the union is a whole space X) of regular (in some sense) families from $\mathcal{I}$ are equal to $\mathfrak{c}$. The last property can be used to obtain some ZFC results about nonmeasurability of unions of small subsets of Polish space.

Here we say that a subset $A \subseteq X$ is a completely $\mathcal{I}$-nonmeasurable if for any $B \in Bor(X) \setminus \mathcal{I}$ we have $A \cap B \neq \emptyset$ and $A^c \cap B \neq \emptyset$.

- If $\mathcal{I} = [X]^{\leq\omega}$ then completely $\mathcal{I}$-nonmeasurability of $A \subseteq X$ is equivalent to A is a Bernstein set.
- If $\mathcal{I} = \mathcal{N}([0,1])$ then $A \subseteq [0,1]$ is completely $\mathcal{N}$-nonmeasurable set iff $\lambda_*(A) = 0$ and $\lambda^*(A) = 1$ (where λ_*, λ^* denotes inner and outer Lebesgue measure respectively).
- If $\mathcal{I} = \mathcal{M}$ then $A \subseteq X$ is completely $\mathcal{M}$-nonmeasurable iff A does not have Baire property in each nonempty open subset of X.

Theorem. *Let X and Y be a Polish space and $\mathcal{I}$ be an c.c.c. σ-ideal with a Borel base. Let $F \subseteq X \times Y$ be an analytic relation such that*

- $X \setminus \{x \in X : \ (\exists y \in Y)\ ((x,y) \in F)\} \in \mathcal{I}$,
- $(\forall y \in Y)\ (\{x \in X : \ (x,y) \in F\} \in \mathcal{I})$,
- $(\forall x \in X)\ (|\{y \in Y : \ (x,y) \in F\}| < \aleph_0)$.

Then there exists $Z \subseteq Y$ such that $\{x \in X : (\exists y \in Z) : (x, y) \in F\}$ is completely $\mathcal{I}$-nonmeasurable in X.

Theorem. *Let X be a Polish space and be an $\mathcal{I}$ σ-ideal with Borel base with the following property*

$$(\forall B \in Bor(X) \setminus \mathcal{I})(\exists P \in Perf(X) \setminus \mathcal{I})(P \subseteq B).$$

Let $\mathcal{A} \subseteq \mathcal{I}$ be a partition of X such that

$$(\forall P \in Perf(X)) \left(\bigcup \{A \in \mathcal{A} : A \cap F \neq \emptyset\} \in Bor(X)\right).$$

Then there exists subfamily $\mathcal{A}' \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A}'$ is completely $\mathcal{I}$-nonmeasurable set in space X.

[1] Rałowski, R., Żeberski, Sz., Complete nonmeasurability in regular families. Houston Journal of Mathematics, Vol. 34 No. 3 (2008), 773–780.