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Large cardinals, small uncountable cardinals, and locally compact spaces of countable extent

A *space of countable extent*, also called an ω_1*-compact space*, is one in which every closed discrete subspace is countable.

Theorem. *[2] In MM(S)[S] models, every locally compact, hereditarily normal, ω_1-compact space is σ-countably compact, i.e., the union of countably many countably compact subspaces.*

In stark contrast, we also have:

Theorem. *[2] If $\clubsuit$, then there exists a locally compact, locally countable (hence first countable) ω_1-compact, monotonically normal space of cardinality $\aleph_1$ that is not σ-countably compact.*

MM(S)[S] models require large cardinal axioms, whereas $\clubsuit$ does not, and monotonically normal spaces are hereditarily collectionwise normal and hereditarily countably paracompact. The questions suggested by the following problem are all unanswered:

Problem 1. *Which of the numerous independence results implied by these two theorems requires large cardinal axioms?*

The following related problems are also unsolved:

Problem 2. *Is there a ZFC example of a locally compact, ω_1-compact space of cardinality $\aleph_1$ that is not σ-countably compact? one that is normal?*

More generally, there is the question of what is the minimum cardinality of such spaces. Examples of cardinality $\mathfrak{c}$ were obtained in 1975 by Erik van Douwen [1] but the following improvement ($\mathfrak{b}$ instead of $\mathfrak{c}$) seems to be new:

Theorem. *[3] Every stationary, co-stationary subset of ω_1 has a locally compact normal quasi-perfect preimage of cardinality $\mathfrak{b}$.*

Quasi-perfect functions inversely preserve both ω_1-compactness and lack of σ-countable compactness, and every such subset of ω_1 has both properties.

[1] van Douwen, E.K., A technique for constructing honest locally compact submetrizable examples, Topology Appl. 47 (1992), no. 3, 179–201.

[2] Nyikos, P., Locally compact, ω_1-compact spaces, submitted to Proceedings AMS.

[3] Nyikos, P., The structure of locally compact normal spaces: some quasi-perfect preimages, submitted to Topology and its Applications.