Union theorems for trees

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Joint work with K. Tyros

Outline

Part I: Classical Union Theorems

(1) Folkman Theorem

(2) Carlson-Simpson Theorem

(3) Dual Ramsey Theorem

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Part II: Ramsey Theory of Trees

(4) Halpern-Läuchli Theorem

(5) Dense-set version

(6) Strong-subtree version

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Part III: Dual Ramsey Theory of Trees

(7) Hales-Jewett Theorem for Trees

(8) Union Theorem for Trees

(9) Union Theorem for Trees in Dimension > 1

(10) Conjectures

Part I: Finite Union Theorem

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Theorem (Folkman, 1969)

For every pair of positive integers k and c there is integer $F = F(k, c)$ such that for every c-coloring of the power-set $\mathcal{P}(X)$ of some set X of cardinality $\geq F$, there is a family $\mathbf{D} = (D_i)_{i=1}^{k}$ of pairwise disjoint nonempty subsets of X such that the family

$$\mathcal{U}(\mathbf{D}) = \{\bigcup_{i \in I} D_i : \emptyset \neq I \subseteq \{1, 2, ..., k\}\}$$

of unions is monochromatic.

Infinite Union Theorem

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Theorem (Carlson-Simpson, 1984)

For every finite Souslin measurable coloring of the power-set $\mathcal{P}(\omega)$ of ω, there is a sequence $\mathbf{D} = (D_n)_{n<\omega}$ of pairwise disjoint nonempty subsets of the natural numbers such that the set

$$\mathcal{U}(\mathbf{D}) = \Big\{ \bigcup_{n \in M} D_n : \ M \text{ is a non-empty subset of } \omega \Big\}$$

is monochromatic.

Dual Ramsey Theorem

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Theorem (Carlson-Simpson, 1984)

For every finite Souslin-measurable coloring of the collection

$$\mathcal{U}^{[\infty]} = \mathcal{U}^{[\infty]}(\omega)$$

of all **infinite** *families of pairwise disjoint nonempty subsets of* ω*, there is a family* $\mathbf{D} = \{D_n : n < \omega\} \in \mathcal{U}^{[\infty]}$ *such that*

$$\mathcal{U}^{[\infty]} \restriction \mathbf{D} = \{\{E_n : n < \omega\} \in \mathcal{U}^{[\infty]} : (\forall n < \omega) E_n \in \mathcal{U}(\mathbf{D})\}$$

is monochromatic.

Part II: Halpern-Läuchli Theorem

A **tree** is a partially ordered set $(T, \leq_T)$ such that

$$\mathrm{Pred}_t(T) = \{s \in T : s <_T t\}$$

is is finite and totally ordered.

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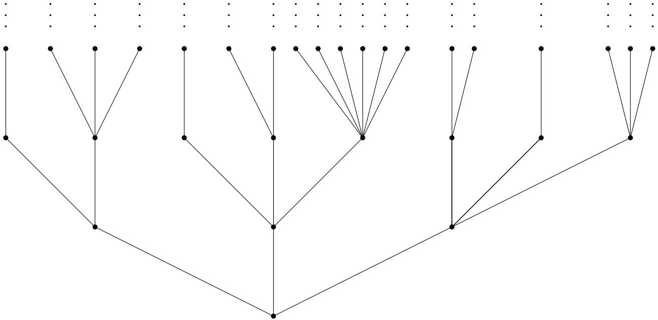
We consider only **rooted and finitely branching trees with no maximal nodes**.



For $n < \omega$, the ***n*th level of** T, is the set

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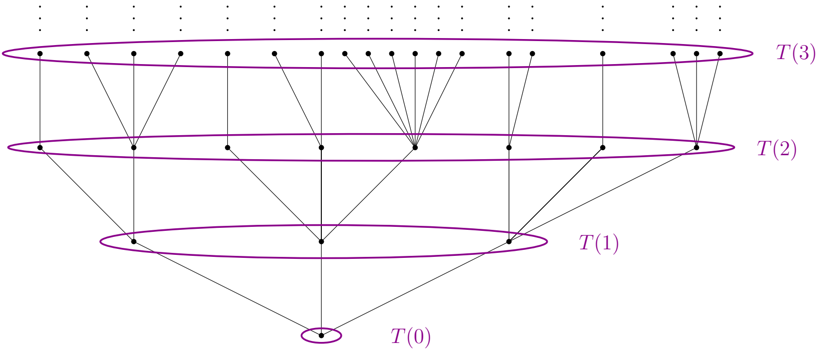

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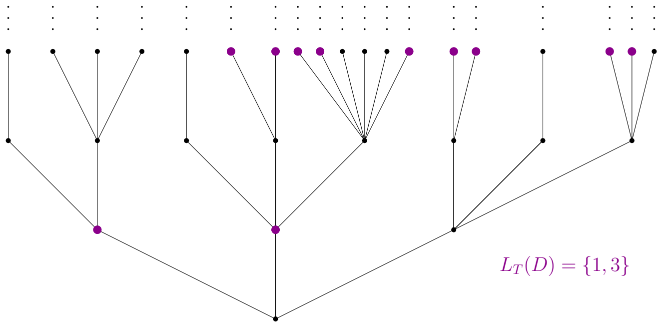

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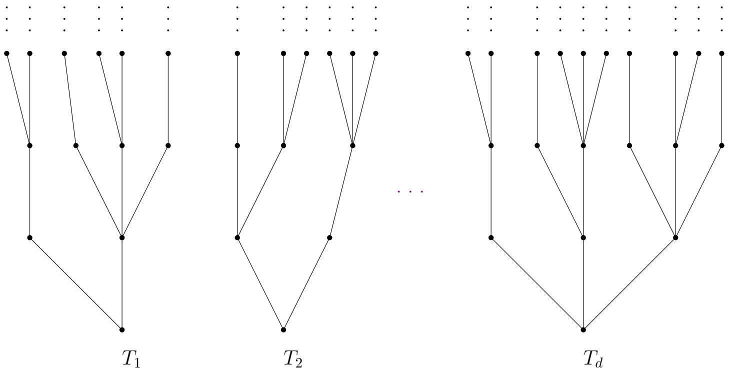

For a vector tree $\mathbf{T} = (T_1, \ldots, T_d)$ we define its **level product** as

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The n**-th level of the level product** of $\mathbf{T}$ is

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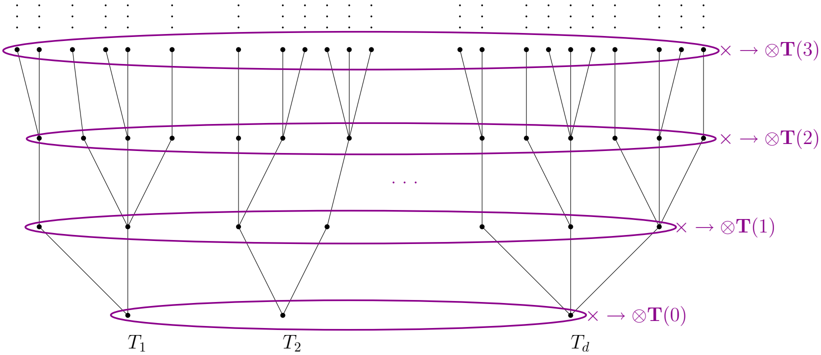

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For $\mathbf{t} = (t_1, ..., t_d)$ in $\otimes\mathbf{T}$, we define

$$\mathrm{Succ}_{\mathbf{t}}(\mathbf{T}) = \{\mathbf{s} \in \otimes\mathbf{T} : \mathbf{t} \leq_{\mathbf{T}} \mathbf{s}\}$$

A sequence $\mathbf{D} = (D_1, ..., D_d)$ is called a **vector subset** of $\mathbf{T}$ if

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For $\mathbf{t} \in \otimes\mathbf{T}$, a vector subset $\mathbf{D}$ of $\mathbf{T}$ is **t-dense** ,

$$(\forall n)(\exists m)(\forall \mathbf{s} \in \otimes\mathbf{T}(n) \cap \mathrm{Succ}_{\mathbf{T}}(\mathbf{t})(\exists \mathbf{s}' \in \otimes\mathbf{T}(m) \cap \otimes\mathbf{D}) \;\; \mathbf{s} \leq_{\mathbf{T}} \mathbf{s}'.$$

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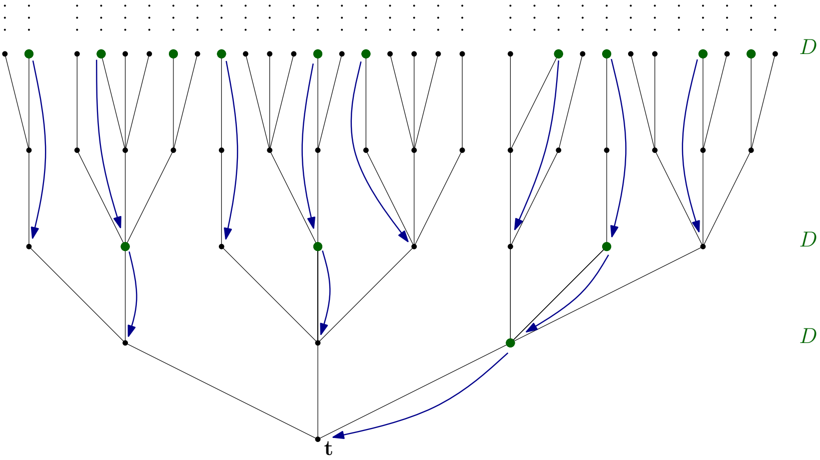
$\mathbf{D}$ is called **dense** if it is $\mathrm{root}(\otimes\mathbf{T})$**-dense**.



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Theorem (Halpern–Läuchli, 1966)

Let $\mathbf{T}$ *be a vector tree. Then for every dense vector subset* $\mathbf{D}$ *of* $\mathbf{T}$ *and every subset* $\mathcal{P}$ *of* $\otimes\mathbf{D}$*, there exists a vector subset* $\mathbf{D}'$ *of* $\mathbf{D}$ *such that either*

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(ii) $\otimes\mathbf{D}'$ *is a subset of* $\mathcal{P}^c$ *and* $\mathbf{D}'$ *is a* $\mathbf{t}$*-dense vector subset* $\mathbf{D}'$ *of* $\mathbf{T}$ *for some* $\mathbf{t}$ *in* $\otimes\mathbf{T}$*.*

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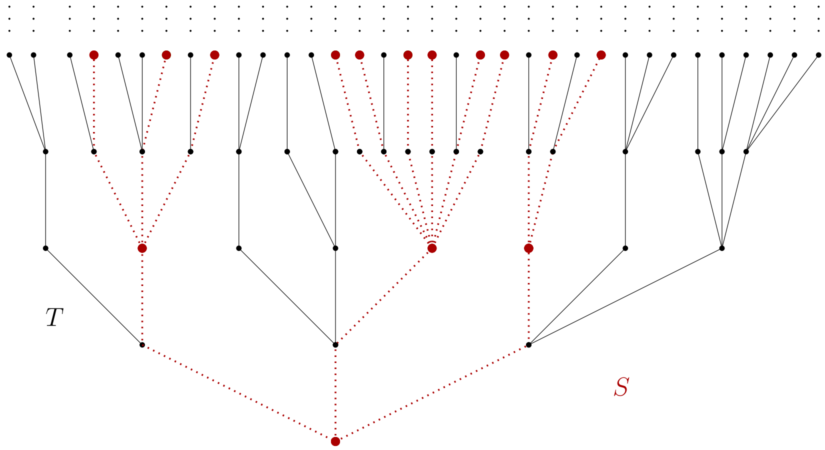
2. Every level of S is subset of some level of T,
3. For every s in S and t' in $\mathrm{ImmSucc}_T(s)$ there is unique s' in $\mathrm{ImmSucc}_S(s)$ with $t \leq_T s'$.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Theorem (Strong Subtree Version of HL)

Let $\mathbf{T}$ *be a vector tree. Then for every finite coloring of* $\otimes\mathbf{T}$ *there exists a vector strong subtree* $\mathbf{S}$ *of* $\mathbf{T}$ *such that* $\otimes\mathbf{S}$ *is monochromatic.*

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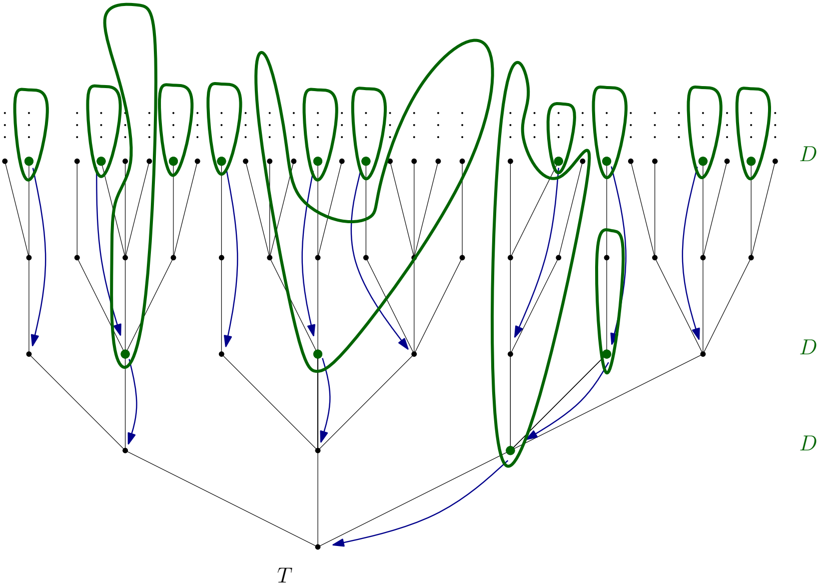
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For a subspace $\mathbf{U} = (U_{\mathbf{t}})_{\mathbf{t} \in \otimes \mathbf{D}(\mathbf{U})}$ we define its **span** by

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If $\mathbf{U}$ and $\mathbf{U}'$ are two subspaces of $\mathcal{U}(\mathbf{T})$, we say that

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Remark

$\mathbf{U}' \leq \mathbf{U}$ implies that $\mathbf{D}(\mathbf{U}')$ is a vector subset of $\mathbf{D}(\mathbf{U})$.

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Let $\mathbf{T}$ *be a vector tree and let* $\mathbf{S}$ *be a vector strong subtree of* $\mathbf{T}$*.*
Let $\mathbf{U}$ *be an* $\mathbf{S}$*-subspace of* $\mathcal{U}(\mathbf{T})$*.*
Then for every finite Souslin measurable coloring of $\mathcal{U}(\mathbf{T})$ *there exist a vector strong subtree* $\mathbf{S}'$ *of* $\mathbf{S}$ *and an* $\mathbf{S}'$*-subspace* $\mathbf{U}'$ *of* $\mathcal{U}(\mathbf{T})$ *with* $\mathbf{U}' \leq \mathbf{U}$ *such that* $[\mathbf{U}']$ *is monochromatic.*

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Corollary

Let $\mathbf{T}$ be a vector tree and let $\mathbf{S}$ be a vector strong subtree of $\mathbf{T}$. Let $\mathbf{U}$ be an $\mathbf{S}$-subspace of $\mathcal{U}(\mathbf{T})$. Then for every finite Souslin measurable coloring of $\mathcal{U}(\mathbf{T})$ there exist a vector strong subtree $\mathbf{S}'$ of $\mathbf{S}$ and an $\mathbf{S}'$-subspace $\mathbf{U}'$ of $\mathcal{U}(\mathbf{T})$ with $\mathbf{U}' \leq \mathbf{U}$ such that $[\mathbf{U}']$ is monochromatic.

Corollary (Carlson-Simpson, 1984)

For every finite Souslin measurable coloring of $\mathcal{P}(\omega)$ there is a sequence $\mathbf{D} = (D_n)_{n<\omega}$ of pairwise disjoint subsets of ω such that the set

$$\mathcal{U}(\mathbf{D}) = \Big\{ \bigcup_{n \in M} D_n : \ M \text{ is a non-empty subset of } \omega \Big\}$$

is monochromatic.

Hales-Jewett Theorem

Theorem (Hales-Jewett, 1963)

*Let Λ be a finite alphabet and let $v \notin \Lambda$ be a variable. Then for every integer $c \geq 1$ there is a number $HJ(\Lambda, c)$ such that for every integer $N \geq HJ(\Lambda, c)$ and every c-coloring of the set of Λ-***words*** *of length N, i.e., the cube Λ^N there is a* **variable word** *$x(v)$ of length N, an element of $(\Lambda \cup \{v\})^N \setminus \Lambda^N$ such that the set of all* **substitutions**

$$\{x[\lambda] : \lambda \in \Lambda\}$$

is monochromatic.

Hales-Jewett Theorem

Theorem (Hales-Jewett, 1963)

*Let Λ be a finite alphabet and let $v \notin \Lambda$ be a variable. Then for every integer $c \geq 1$ there is a number $HJ(\Lambda, c)$ such that for every integer $N \geq HJ(\Lambda, c)$ and every c-coloring of the set of Λ-***words*** *of length N, i.e., the cube Λ^N there is a* **variable word** *$x(v)$ of length N, an element of $(\Lambda \cup \{v\})^N \setminus \Lambda^N$ such that the set of all* **substitutions**

$$\{x[\lambda] : \lambda \in \Lambda\}$$

is monochromatic.

Theorem (Carlson-Simpson, 1984)

Let Λ be a finite alphabet and let $v \notin \Lambda$ be a variable. Then for every finite coloring of the **semigroup** *W_Λ of all Λ-words, there is an* **infinite sequence** *$(x_n(v))$* **of variable words** *such that the set*

$$\{x_0[\lambda_0]^\frown \cdots {}^\frown x_n[\lambda_n] : n < \omega, \lambda_0, ..., \lambda_n \in \Lambda\}$$

is monochromatic.

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For $m < n < \omega$, set

$$\mathrm{W}(\Lambda, \mathbf{T}, m, n) = \Lambda^{\otimes \mathbf{T} \restriction [m,n)},$$

where $\otimes \mathbf{T} \restriction [m, n) = \bigcup_{j=m}^{n-1} \otimes \mathbf{T}(j)$.

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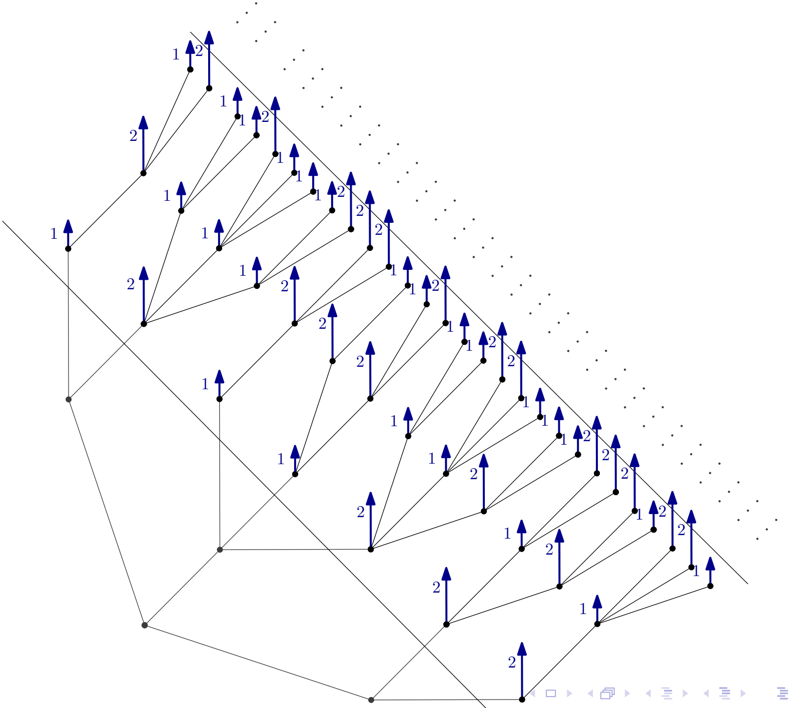
where $\otimes \mathbf{T} \upharpoonright [m, n) = \bigcup_{j=m}^{n-1} \otimes \mathbf{T}(j)$.We also set

$$\mathrm{W}(\Lambda, \mathbf{T}) = \bigcup_{m \leq n} \mathrm{W}(\Lambda, \mathbf{T}, m, n).$$



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to be the set of all functions

$$f : \otimes\mathbf{T} \restriction [m, n) \to \Lambda \cup \{v_{\mathbf{s}} : \mathbf{s} \in \otimes\mathbf{D}\}$$

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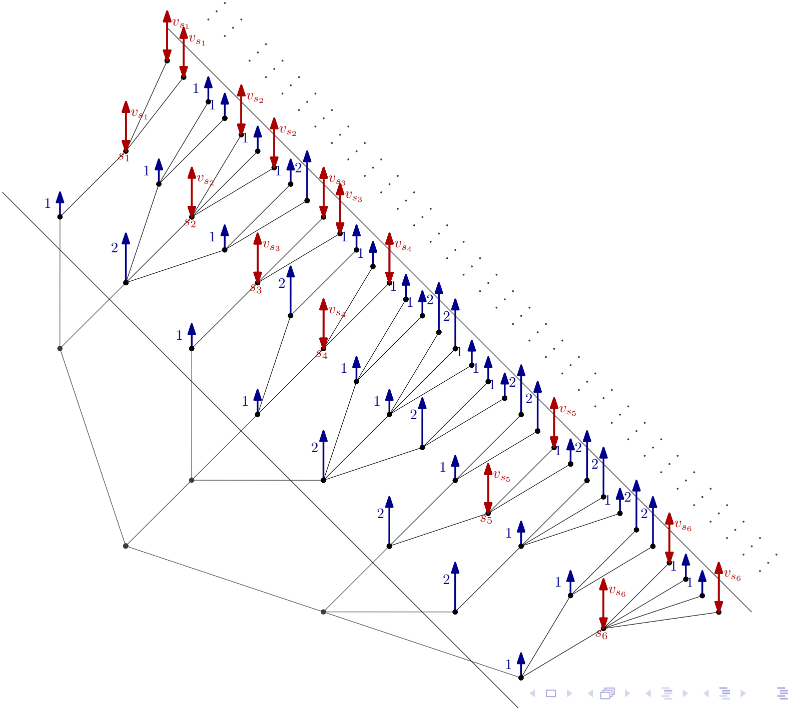

v_{s_1}
v_{s_1}
v_{s_1}
s_1
v_{s_2}
v_{s_2}
v_{s_2}
s_2
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v_{s_3}
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s_3
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v_{s_5}
s_5
v_{s_6}
v_{s_6}
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s_6

For $f \in \mathrm{W}_v(\Lambda, \mathbf{T}, \mathbf{D}, m, n)$, set

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Moreover, we set

$$\mathrm{W}_v(\Lambda, \mathbf{T}) = \bigcup \Big\{ \mathrm{W}_v(\Lambda, \mathbf{T}, \mathbf{D}, m, n) : m \leq n \text{ and } \mathbf{D} \text{ is a vector level subset of } \mathbf{T} \text{ with } L_{\mathbf{T}}(\mathbf{D}) \subset [m, n) \Big\}.$$

The elements of $\mathrm{W}_v(\Lambda, \mathbf{T})$ are viewed as **variable words over the alphabet** Λ**.**

For variable words f in $\mathrm{W}_v(\Lambda, \mathbf{T})$ we take **substitutions**:
For every family $\mathbf{a} = (a_{\mathbf{s}})_{\mathbf{s} \in \otimes \mathrm{ws}(f)} \subseteq \Lambda$, let
$f(\mathbf{a}) \in \mathrm{W}(\Lambda, \mathbf{T})$ be the result of substituting for every $\mathbf{s}$ in $\otimes \mathrm{ws}(f)$ each occurrence of $v_{\mathbf{s}}$ by $a_{\mathbf{s}}$, .

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Moreover, we set

$$[f]_\Lambda = \{f(\mathbf{a}) : \ \mathbf{a} = (a_{\mathbf{s}})_{\mathbf{s} \in \otimes \mathrm{ws}(f)} \subseteq \Lambda\},$$

the constant span of f.

Subspaces

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For a subspace $X = (f_n)_{n<\omega}$ we define

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For two subspaces X and Y, we write $X \leq Y$ if $[X]_\Lambda \subseteq [Y]_\Lambda$.

An infinite Hales-Jewett theorem for trees

Theorem

Let Λ be a finite alphabet and $\mathbf{T}$ a vector tree. Then for every finite coloring of the set of the constant words $\mathrm{W}(\Lambda, \mathbf{T})$ over Λ and every subspace X of $\mathrm{W}(\Lambda, \mathbf{T})$ there exists a subspace X' of $\mathrm{W}(\Lambda, \mathbf{T})$ with $X' \leq X$ such that the set $[X']_\Lambda$ is monochromatic.

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Remark

This will be used as a **pigeonhole principle** for its infinite-dimensional version.

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For a subspace X, we set

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Higher Dimensions

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Theorem (Graham-Rothschild)

For every triple of positive integers k, l, and c there is integer $GR = GR(k, l, c)$ such that for every set X of cardinality $\geq GR$ and every c-coloring of the family

$$\binom{\mathcal{P}(X)}{k}$$

of all k-families of pairwise disjoint subsets of X there is a family $\mathbf{D} = (D_i)_{i=1}^{l}$ of pairwise disjoint nonempty subsets of X such that the family

$$\binom{\mathcal{U}(\mathbf{D})}{k}$$

of k-families of pairwise disjoint subsets of $\mathcal{U}(\mathbf{D}) = \{\bigcup_{i \in I} D_i : \emptyset \neq I \subseteq \{1, 2, ..., l\}\}$ is monochromatic.

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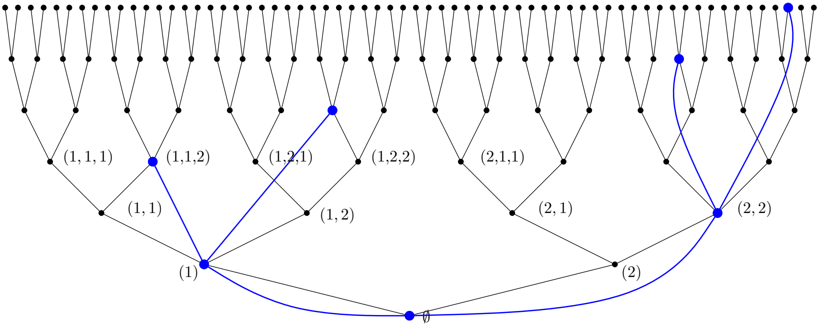
4. For every $\ell < k$ and $s, t \in T(\ell)$, we have that $s <_{\mathrm{lex}} t$ iff $|s| < |t|$.



Skew subtree of height 3

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A subspace $(V_s)_{s \in S}$ is a **further subspace of** $(U_t)_{t \in T}$ if

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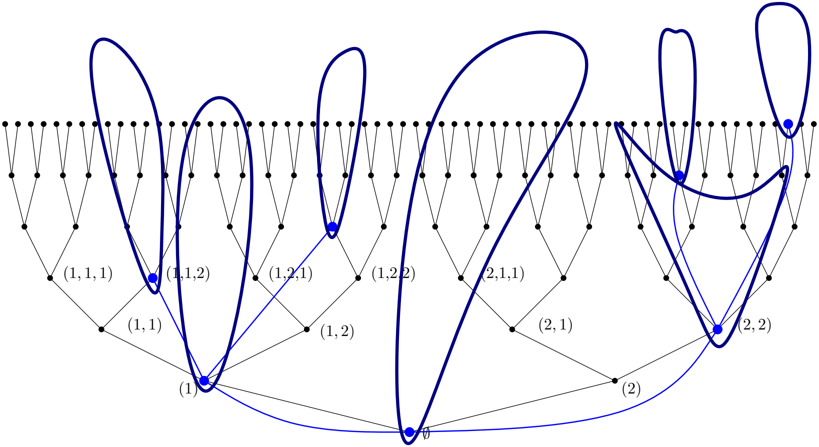
(1, 1, 1)
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(1)
(2)
∅

Finite Union Theorem for Trees in Dimension k

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Theorem

For every positive integers c, k, l, b with $k \leq l$ there exists a positive integer $n_0 = TT(c, k, l, b)$ such that for every integer $n \geq n_0$ and every r-coloring of the k-dimensional subspaces of $\mathcal{U}(b^{<n})$, there exists a l-dimensional subspace $\mathbf{U}$ such that the set of all further k-dimensional subspaces of $\mathbf{U}$ is monochromatic.

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Remark

The Graham-Rotschild Finite Union Theorem is the case $b = 1$ of this result.

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As before an ω-dimensional subspace $(V_s)_{s\in S}$ is a **further subspace of** $(U_t)_{t\in T}$ if

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Conjecture

For every finite Souslin-measurable coloring of the family of all ω-dimensional subspaces of $\mathcal{U}(b^{<\omega})$ there is an ω-dimensional subspace $(U_t)_{t \in T}$ all of whose further ω-dimensional subspaces are of the same color.