NUMBER THEORY IN THE STONE-ČECH COMPACTIFICATION

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The Stone-Čech compactification

S - discrete topological space

βS - the set of ultrafilters on S

Base sets: $\bar{A} = \{p \in \beta S : A \in p\}$ for $A \subseteq S$

Principal ultrafilters $\{A \subseteq S : n \in A\}$ are identified with respective elements $n \in S$

$S^* = \beta S \setminus S$

If $A \in [S]^{\aleph_0}$ we think of βA as a subspace of βS

If C is a compact topological space, every (continuous) function $f : S \to C$ can be extended uniquely to $\tilde{f} : \beta S \to C$

In particular, every function $f : S \to S$ can be extended uniquely to $\tilde{f} : \beta S \to \beta S$

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Algebra in the Stone-Čech compactification

$(S,\cdot)$ - a semigroup provided with discrete topology

For $A \subseteq S$ and $n \in S$:

$$A/n = \{m \in S : mn \in A\}$$

The semigroup operation can be extended to βS as follows:

$$A \in p \cdot q \Leftrightarrow \{n \in S : A/n \in q\} \in p.$$

Theorem (HS)

(a) $(\beta S,\cdot)$ is a semigroup.
(b) If $S = N$, the algebraic center $\{p \in \beta N : \forall x \in \beta N \; px = xp\}$ of $(\beta N,\cdot)$ is N.

[HS] Hindman, Strauss: *Algebra in the Stone-Čech compactification, theory and applications*

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The natural numbers

The idea: work with $S = N$ and translate problems in number theory to $(\beta N, \cdot)$

Example

Problem: are there infinitely many perfect numbers?

$n \in N$ is perfect if $\sigma(n) = 2n$, where $\sigma(n)$ is the sum of positive divisors of n.

If the answer is "yes", then there is $p \in N^*$ such that $\{n \in N : \sigma(n) = 2n\} \in p$, so $\tilde{\sigma}(p) = 2p$.

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Extensions of the divisibility relation

Definition

Let $p, q \in \beta N$.
(a) q is left-divisible by p, $p \mid_L q$, if there is $r \in \beta N$ such that $q = rp$.
(b) q is right-divisible by p, $p \mid_R q$, if there is $r \in \beta N$ such that $q = pr$.
(c) q is mid-divisible by p, $p \mid_M q$, if there are $r, s \in \beta N$ such that $q = rps$.

Clearly, $\mid_L \subseteq \mid_M$ and $\mid_R \subseteq \mid_M$.

Lemma

No two of the relations $\mid_L$, $\mid_R$ and $\mid_M$ are the same.

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Continuity of $|_R$

A binary relation $\alpha \subseteq X^2$ is *continuous* if for every open set $U \subseteq X$ the set $\alpha^{-1}[U] = \{x \in X : \exists y \in U\ (x, y) \in \alpha\}$ is also open.

Lemma

The relation $|_R$ is a continuous extension of $|$ to βN.

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The relation $|_R$ is a continuous extension of $|$ to βN.

Divisibility by elements of N

Theorem (HS)

N^* is an ideal of βN.

For $n \in N$ and $p \in \beta N$, $n \mid_L p$ iff $n \mid_R p$ iff $n \mid_M p$, so we write only $n \mid p$.

Lemma

If $n \in N$, $n \mid p$ if and only if $nN \in p$.

Theorem

Let $A \subseteq N$ be downward closed for $\mid$ and closed for the operation of least common multiple. Then there is $x \in \beta N$ divisible by all $n \in A$, and not divisible by any $n \notin A$.

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Prime and irreducible elements

An element $p \in \beta N$ is *irreducible* in $X \subseteq \beta N$ if it can not be represented in the form $p = xy$ for $x, y \in X \setminus \{1\}$.
$p \in \beta N$ is *prime* if $p \mid_R xy$ for $x, y \in \beta N$ implies $p \mid_R x$ or $p \mid_R y$.

Lemma

If $n \in N$ is a prime number and $n \mid xy$ for some $x, y \in \beta N$, then $n \mid x$ or $n \mid y$.

Let $P = \{n \in N : n \text{ is prime}\}$

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If $p \in \beta N$ and $P \in p$, then p is irreducible in βN.

The reverse is not true: there is $p \in \beta N$ irreducible in βN such that $P \notin p$.

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Prime and irreducible elements (continued)

Theorem (HS)

N^*N^* is nowhere dense in N^*, i.e. for every $A \in [N]^{\aleph_0}$ there is $B \in [A]^{\aleph_0}$ such that all elements of $\bar{B}$ are irreducible in N^*.

$K(\beta N)$ - the smallest ideal of βN

Theorem (HS)

The following conditions are equivalent: (i) $p \in K(\beta N)$ (ii) $p \in \beta N qp$ for all $q \in \beta N$ (iii) $p \in pq\beta N$ for all $q \in \beta N$.

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Cancelation laws

Theorem (HS)

If $n \in N$ and $p, q \in \beta N$, then $np = nq$ implies $p = q$.

Theorem (HS)

If $m, n \in N$ and $p \in \beta N$, then $mp = np$ implies $m = n$.

Theorem (Blass, Hindman)

$p \in \beta N$ is right cancelable if and only if for every $A \subseteq N$ there is $B \subseteq A$ such that $A = \{x \in N : B/x \in p\}$.

Theorem (Blass, Hindman)

The set of right cancelable elements contains an dense open subset of N^*, i.e. for every $U \in [N]^{\aleph_0}$ there is $V \in [U]^{\aleph_0}$ such that all $p \in \bar{V}$ are right cancelable.

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More on $|_L$

Let E_L be the symmetric closure of $|_L$.

Theorem (HS)

Each of the connected components of the graph $(\beta N, E_L)$ is nowhere dense in βN.

Definition

(a) $p \mid_{LN} q$ if there is $n \in N$ such that $p \mid_L nq$
(b) $p =_{LN} q$ if $p \mid_{LN} q$ and $q \mid_{LN} p$.

Lemma

For every $q \in \beta N$ the set $q{\downarrow} = \{[p]_{=_{LN}} : p \mid_{LN} q\}$ is linearly ordered.

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Equivalent conditions for divisibility

For $p \in \beta N$:
$C(p) = \{A \subseteq N : \forall n \in N \ A/n \in p\}$
$D(p) = \{A \subseteq N : \{n \in N : A/n = N\} \in p\}$

Theorem

The following conditions are equivalent: (i) $p \mid_L q$; (ii) $C(p) \subseteq q$; (iii) $C(p) \subseteq C(q)$.

Conjecture: the following conditions are equivalent: (i) $p \mid_R q$; (ii) $D(p) \subseteq q$; (iii) $D(p) \subseteq D(q)$.

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For $p \in \beta N$:
$C(p) = \{A \subseteq N : \forall n \in N \ A/n \in p\}$
$D(p) = \{A \subseteq N : \{n \in N : A/n = N\} \in p\}$

Theorem

The following conditions are equivalent: (i) $p \mid_L q$; (ii) $C(p) \subseteq q$; (iii) $C(p) \subseteq C(q)$.

Conjecture: the following conditions are equivalent: (i) $p \mid_R q$; (ii) $D(p) \subseteq q$; (iii) $D(p) \subseteq D(q)$.

Equivalent conditions for divisibility

For $p \in \beta N$:
$C(p) = \{A \subseteq N : \forall n \in N\ A/n \in p\}$
$D(p) = \{A \subseteq N : \{n \in N : A/n = N\} \in p\}$

Theorem

The following conditions are equivalent: (i) $p \mid_L q$; (ii) $C(p) \subseteq q$; (iii) $C(p) \subseteq C(q)$.

Conjecture: the following conditions are equivalent: (i) $p \mid_R q$; (ii) $D(p) \subseteq q$; (iii) $D(p) \subseteq D(q)$.