A remark on the general nature of the Katětov's construction

Dragan Mašulović

Department of Mathematics and Informatics
University of Novi Sad, Serbia

joint work with Wiesław Kubiś

SE⊨OP 2014, Novi Sad, 18 Aug 2014

The Urysohn space

P. URYSOHN: *Sur un espace métrique universel.*

Bull. Math. Sci. 51 (1927), 43–64, 74–90

U — complete separable metric space which is homogeneous and embeds all separable metric spaces.

$$U = \overline{U_{\mathbb{Q}}}$$

Katětov's construction of the Urysohn space

M. KATĚTOV: *On universal metric spaces.*

General topology and its relations to modern analysis and algebra. VI (Prague, 1986), Res. Exp. Math. vol. 16, Heldermann, Berlin, 1988, 323–330

A *Katětov function over a finite rational metric space X* is every function $\alpha : X \to \mathbb{Q}$ such that

$$|\alpha(x) - \alpha(y)| \leqslant d(x, y) \leqslant \alpha(x) + \alpha(y)$$

$K(X) =$ all Katětov functions over X, which is a rational metric space under sup metric

$$\operatorname{colim}(X \hookrightarrow K(X) \hookrightarrow K^2(X) \hookrightarrow K^3(X) \hookrightarrow \cdots) = U_{\mathbb{Q}}$$

Katětov's construction of the Urysohn space

M. KATĚTOV: *On universal metric spaces.*

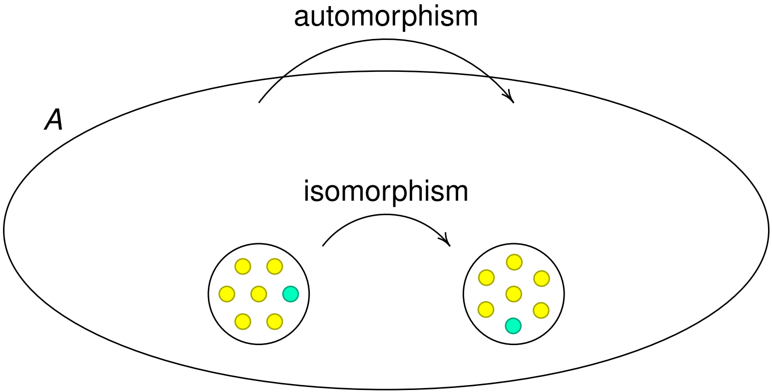
General topology and its relations to modern analysis and algebra. VI (Prague, 1986), Res. Exp. Math. vol. 16, Heldermann, Berlin, 1988, 323–330

Observation 1. $U_{\mathbb{Q}}$ is countable and homogeneous.

Observation 2. $K(X)$ contains all 1-point extensions of X.

Observation 3. K is functorial.

Homogeneity



Fraïssé theory

age(A) — the class of all finitely generated struct's which embed into A

amalgamation class — a class $\mathcal{K}$ of fin. generated struct's s.t.

- there are countably many pairwise noniso struct's in $\mathcal{K}$;
- $\mathcal{K}$ has (HP);
- $\mathcal{K}$ has (JEP); and
- $\mathcal{K}$ has (AP):

  for all $A, B, C \in \mathcal{K}$ and embeddings $f : A \hookrightarrow B$ and $g : A \hookrightarrow C$, there exist $D \in \mathcal{K}$ and embeddings $u : B \hookrightarrow D$ and $v : C \hookrightarrow D$ such that $u \circ f = v \circ g$.

$$\begin{array}{ccc} C & \overset{v}{\hookrightarrow} & D \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle u} \\ A & \underset{f}{\hookrightarrow} & B \end{array}$$

Fraïssé theory

Theorem. *[Fraisse, 1953]*

1. If A is a countable homogeneous structure, then **age**(A) is an amalgamation class.
2. If $\mathcal{K}$ is an amalgamation class, then there is a unique (up to isomorphism) countable homogeneous structure A such that **age**(A) $= \mathcal{K}$.
3. If B is a countable structure *younger than* A (that is, **age**(B) $\subseteq$ **age**(A)), then $B \hookrightarrow A$.

Definition. If $\mathcal{K}$ is an amalgamation class and A is the countable homogeneous structure such that **age**(A) $= \mathcal{K}$, we say that A is the *Fraïssé limit* of $\mathcal{K}$ and write $A = \text{Flim}(\mathcal{K})$.

Some prominent Fraïssé limits

$\mathbb{Q}$ — the Fraïssé limit of the class of all linear orders

$U_{\mathbb{Q}}$ — the Fraïssé limit of the class of finite metric spaces with rational distances (the rational Urysohn space)

R — the Fraïssé limit of the class of all finite graphs (the Rado graph)

H_n — the Fraïssé limit of the class of all finite K_n-free graphs, $n \geqslant 3$ (Henson graphs)

P — the Fraïssé limit of the class of all finite posets (the random poset)

Recall:

M. KATĚTOV: *On universal metric spaces.*

General topology and its relations to modern analysis and algebra. VI (Prague, 1986), Res. Exp. Math. vol. 16, Heldermann, Berlin, 1988, 323–330

Katětov's construction

$$\operatorname{colim}(X \hookrightarrow K(X) \hookrightarrow K^2(X) \hookrightarrow K^3(X) \hookrightarrow \cdots) = U_{\mathbb{Q}}$$

Observation 1. $U_{\mathbb{Q}}$ is countable and homogeneous.

Observation 2. $K(X)$ contains all 1-point extensions of X.

Observation 3. K is functorial.

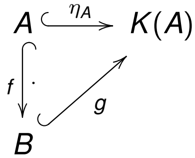
Katětov functors

$\mathcal{A}$ — a category of fin generated *L*-struct's with (HP) and (JEP)

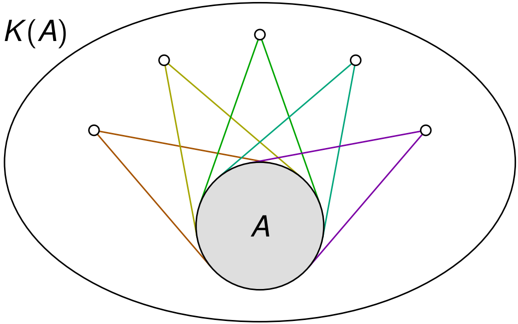
$\mathcal{C}$ — the category of all colimits of ω-chains in $\mathcal{A}$

Definition. A functor $K : \mathcal{A} \to \mathcal{C}$ is a *Katětov functor* if

1. K preserves embeddings, and
2. there exists a natural transformation $\eta : \mathrm{ID} \to K$ such that for every embedding $f : A \hookrightarrow B$ in $\mathcal{A}$ where B is a 1-point extension of A there is an embedding $g : B \hookrightarrow K(A)$ satisfying $\nearrow$



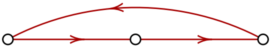
Katětov functors



$K(A)$ is “a functorial amalgam” of all 1-point extensions of A.

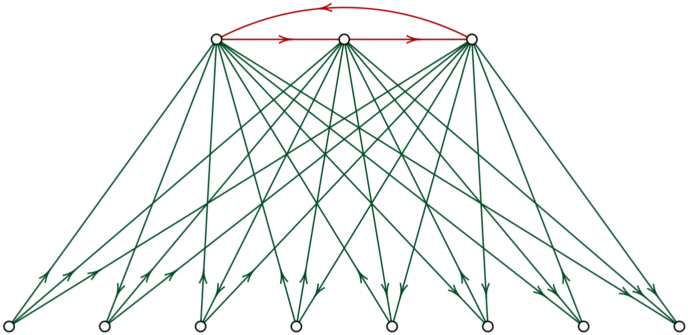
Why is it hard to construct a Katětov **functor** by hand?

Example. Tournaments.



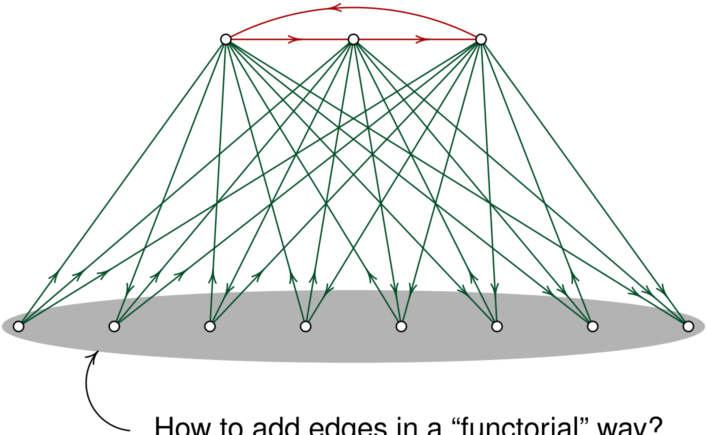
Why is it hard to construct a Katětov **functor** by hand?

Example. Tournaments.



Why is it hard to construct a Katětov **functor** by hand?

Example. Tournaments.



How to add edges in a “functorial” way?

Why is it hard to construct a Katětov **functor** by hand?

Example. Tournaments.

$T = (V, E)$ — a tournament with n vertices
$T^{\leqslant n}$ — the tournament with vertices $V^{\leqslant n}$ and edges defined by:

- if s and t are seq's such that $|s| < |t|$, put $s \to t$ in $T^{\leqslant n}$;
- if $s = \langle s_1, \dots, s_k \rangle$ and $t = \langle t_1, \dots, t_k \rangle$ are distinct sequences of the same length, find the smallest i such that $s_i \neq t_i$ and then put $s \to t$ in $T^{\leqslant n}$ if and only if $s_i \to t_i$ in T.

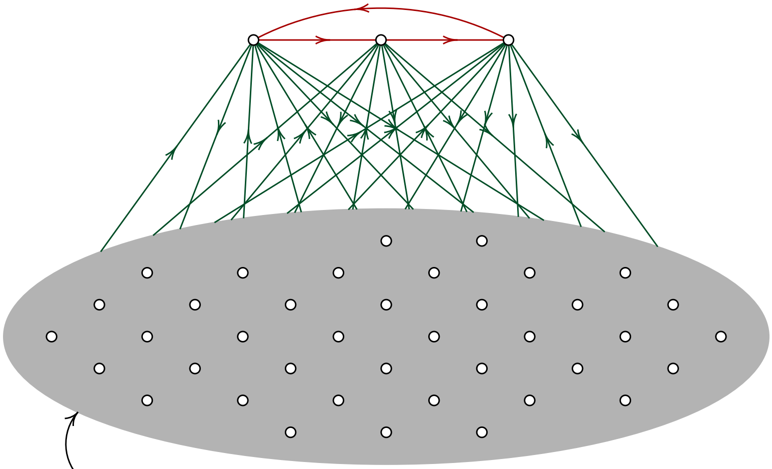
Put $K(T) = (V^*, E^*)$ where

$$V^* = V \cup V^{\leqslant n},$$

$$E^* = E \cup E(T^{\leqslant n}) \cup \{v \to s : v \in V, s \in V^{\leqslant n}, v \text{ appears in } s\}$$
$$\cup \{s \to v : v \in V, s \in V^{\leqslant n}, v \text{ does not appear in } s\}.$$

Why is it hard to construct a Katětov **functor** by hand?

Example. Tournaments.



approx. $2n^n$ new vertices

Katětov functors

$\mathcal{A}$ — a category of fin generated *L*-struct's with (HP) and (JEP)

$\mathcal{C}$ — the category of all colimits of ω-chains in $\mathcal{A}$

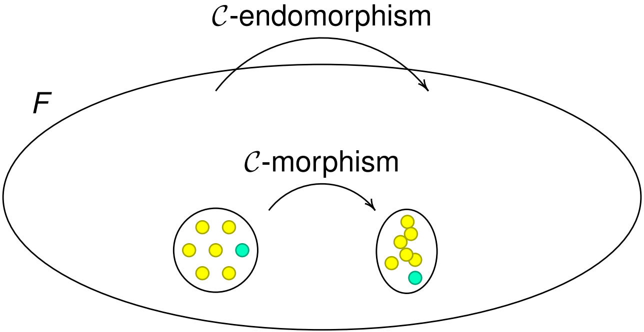
Theorem. If there exists a Katětov functor $K : \mathcal{A} \to \mathcal{C}$, then

1. $\mathcal{A}$ is an amalgamation class,
2. its Fraïssé limit F can be obtained by the "Katětov construction" starting from an arbitrary $A \in \mathcal{A}$:

$$F = \mathrm{colim}(A \hookrightarrow K(A) \hookrightarrow K^2(A) \hookrightarrow K^3(A) \hookrightarrow \cdots),$$

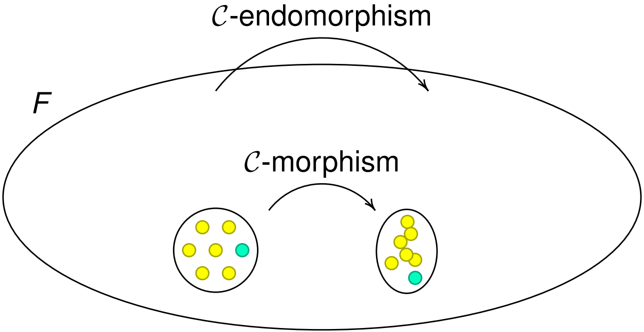
3. F is $\mathcal{C}$*-morphism-homogeneous*.

$\mathcal{C}$-morphism-homogeneity



Definition. A structure F is $\mathcal{C}$-morphism-homogeneous if every $\mathcal{C}$-morphism between finitely induced substructures of F extends to a $\mathcal{C}$-endomorphism of F.

$\mathcal{C}$-morphism-homogeneity



P. J. CAMERON, J. NEŠETŘIL: *Homomorphism-homogeneous relational structures.* Combin. Probab. Comput, 15 (2006), 91–103

Katětov functors: Examples

A Katětov functor exists for the following categories $\mathcal{A}$:

- finite linear orders with order-preserving maps,
- finite graphs with graph homomorphisms,
- finite K_n-free graphs with embeddings,
- finite digraphs with digraph homomorphisms,
- finite tournaments with homomorphisms = embeddings.
- finite rational metric spaces with nonexpansive maps,
- finite posets with order-preserving maps,
- finite boolean algebras with homomorphisms,
- finite semilattices/lattices/distributive lattices with embeddings.

A Katětov functor **does not exist** for the category of finite K_n-free graphs and graph homomorphisms.

Existence of Katětov functors

$\mathcal{A}$ — a category of fin generated L-struct's with (HP) and (JEP)

$\mathcal{C}$ — the category of all colimits of ω-chains in $\mathcal{A}$

Theorem. There exists a Katětov functor $K : \mathcal{A} \to \mathcal{C}$ *if and only if* $\mathcal{A}$ is an amalgamation class with the *morphism extension property.*

Morphism extension property

$\mathcal{C}$ — a category

Definition. $C \in \mathcal{C}$ has the *morphism extension property in* $\mathcal{C}$ if for any choice $f_1, f_2, \ldots$ of partial $\mathcal{C}$-morphisms of C there exist $D \in \mathcal{C}$ and $m_1, m_2, \ldots \in \mathrm{End}_{\mathcal{C}}(D)$ such that C is a substructure of D, m_i is an extension of f_i for all i, and the following *coherence* conditions are satisfied for all i, j and k:

- if $f_i = \mathrm{id}_A$, $A \leqslant C$, then $m_i = \mathrm{id}_D$,
- if f_i is an embedding, then so is m_i, and
- if $f_i \circ f_j = f_k$ then $m_i \circ m_j = m_k$.

We say that $\mathcal{C}$ has the *morphism extension property* if every $C \in \mathcal{C}$ has the morphism extension property in $\mathcal{C}$.

Existence of Katětov functors for algebras

L — algebraic language

$\mathcal{V}$ — a variety of L-algebras understood as a category of L-algebras with embeddings

$\mathcal{A}$ — the full subcategory of $\mathcal{V}$ spanned by all finitely generated algebras in $\mathcal{V}$

$\mathcal{C}$ — the full subcategory of $\mathcal{V}$ spanned by all countably generated algebras in $\mathcal{V}$

Theorem. There exists a Katětov functor $K : \mathcal{A} \to \mathcal{C}$ *if and only if* $\mathcal{A}$ is an amalgamation class.

The Importance of Being ~~Earnest~~ Functor

Theorem. Let $K : \mathcal{A} \to \mathcal{C}$ be a Katětov functor and let F be the Fraïssé limit of $\mathcal{A}$. Then for every object C in $\mathcal{C}$:

- $\mathrm{Aut}(C) \hookrightarrow \mathrm{Aut}(F)$;
- $\mathrm{End}_{\mathcal{C}}(C) \hookrightarrow \mathrm{End}_{\mathcal{C}}(F)$.

Proof (Idea). Take any $f : C \to C$. Then:

$$
\begin{array}{ccccccccc}
C & \xhookrightarrow{\eta} & K(C) & \xhookrightarrow{\eta} & K^2(C) & \xhookrightarrow{\eta} & \cdots & \rightsquigarrow & F \\
{\scriptstyle f}\downarrow & & {\scriptstyle K(f)}\downarrow & & {\scriptstyle K^2(f)}\downarrow & & & & \downarrow{\scriptstyle f^*} \\
C & \xhookrightarrow[\eta]{} & K(C) & \xhookrightarrow[\eta]{} & K^2(C) & \xhookrightarrow[\eta]{} & \cdots & \rightsquigarrow & F
\end{array}
\qquad \square
$$

The Importance of Being ~~Earnest~~ Functor

Theorem. Let $K : \mathcal{A} \to \mathcal{C}$ be a Katětov functor and let F be the Fraïssé limit of $\mathcal{A}$. Then for every object C in $\mathcal{C}$:

- $\mathrm{Aut}(C) \hookrightarrow \mathrm{Aut}(F)$;
- $\mathrm{End}_{\mathcal{C}}(C) \hookrightarrow \mathrm{End}_{\mathcal{C}}(F)$.

Moreover, if K is *locally finite* (that is, $K(A)$ is finite whenever A is finite), then the above embeddings are countinuous w.r.t. the topology of pointwise convergence.

The Importance of Being ~~Earnest~~ Functor

Corollary. For the following Fraïssé limits F we have that Aut(F) embeds all permutation groups on a countable set:

- $\mathbb{Q}$,
- the random graph [Henson 1971],
- Henson graphs [Henson 1971],
- the random digraph,
- the rational Urysohn space [Uspenskij 1990],
- the random poset,
- the countable atomless boolean algebra,
- the random semilattice,
- the random lattice,
- the random distributive lattice.

The Importance of Being ~~Earnest~~ Functor

Corollary. For the following Fraïssé limits *F* we have that End(*F*) embeds all transformation monoids on a countable set:

- $\mathbb{Q}$,
- the random graph [Bonato, Delić, Dolinka 2010],
- the random digraph,
- the rational Urysohn space,
- the random poset [Dolinka 2007],
- the countable atomless boolean algebra.

The Importance of Being ~~Earnest~~ Functor

$\mathcal{C}$ — a locally finite category of L-struct's and all L-hom's

$\mathcal{A}$ — the full subcategory of $\mathcal{C}$ consisting of all finite struct's in $\mathcal{C}$

Theorem. Assume that there exists a locally finite Katětov functor $K : \mathcal{A} \to \mathcal{C}$. Then the following are equivalent for a $C \in \mathcal{C}$:

1. C is locally K-closed;
2. C is algebraically closed in $\mathcal{C}$;
3. C is a retract of Flim($\mathcal{A}$).

The Importance of Being ~~Earnest~~ Functor

$\mathcal{A}$ — a category of fin generated *L*-struct's with (HP) and (JEP)

$\mathcal{C}$ — the category of all colimits of ω-chains in $\mathcal{A}$

Theorem. Assume that there exists a Katětov functor $K : \mathcal{A} \to \mathcal{C}$ and that $\mathcal{C}$ has *retractive natural (JEP)*. Let *F* be the Fraïssé limit of $\mathcal{A}$. Then:

1. $\mathrm{End}_{\mathcal{C}}(F)$ is *strongly distorted*,
2. the *Sierpiński rank* of $\mathrm{End}_{\mathcal{C}}(F)$ is at most 5,
3. if $\mathrm{End}_{\mathcal{C}}(F)$ is not finitely generated then it has the *semigroup Bergman property*.

The Importance of Being ~~Earnest~~ Functor

Corollary. For the following Fraïssé limits F we have that End(F) has the semigroup Bergman property:

- random graph,
- random digraph,
- rational Urysohn sphere (the Fraïssé limit of the category of all fin met spaces with distances in $[0,1]_{\mathbb{Q}}$),
- random poset,
- random boolean algebra (the Fraïssé limit of the category of all finite boolean algebras).