Discrete subspaces of countably compact spaces

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Introduction

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Any non-isolated point of a compact T_2 space is discretely touchable, i.e. the accumulation point of a discrete set.

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Also, countably tight compact T_2 spaces are discretely generated.

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Consistently, there is an ω-bounded (hence countably compact) regular space with a discretely untouchable point.

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FACT. (van Douwen) There is a countable, crowded, regular space in which every point is discretely untouchable.

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THEOREM (J-Soukup-Szentmiklóssy)

Let X be regular, countably compact, and countably tight. Then for any countable $A \subset \overline{\cup\mathcal{U}} \setminus \cup\mathcal{U}$ there is $D \in \mathcal{I}(S,\mathcal{U})$ s.t. $A \subset \overline{D}$.

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$$\mathcal{I}(S,\mathcal{U}) = \{D \in [S]^{\leq\omega} : \forall\, U \in \mathcal{U}\ \big(|D \cap U| < \omega\big)\}$$

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If X is regular then $\mathcal{I}(S,\mathcal{U})$ is a P-ideal.

THEOREM (J-Soukup-Szentmiklóssy)

Let X be regular, countably compact, and countably tight. Then for any countable $A \subset \overline{\cup\mathcal{U}} \setminus \cup\overline{\mathcal{U}}$ there is $D \in \mathcal{I}(S,\mathcal{U})$ s.t. $A \subset \overline{D}$.

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Regular, countably compact, and countably tight spaces are discretely determined.

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THANK YOU FOR YOUR ATTENTION !