On small expansions of $(\omega, <)$ and $(\omega + \omega^*, <)$

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Definition: We say that T is *binary* if for every formula $\varphi(x_0, \ldots, x_n)$ there is a formula $\psi(x_0, \ldots, x_n)$, a Boolean combination of formulas in at most two variables such that φ and ψ are equivalent modulo T.

Theorem (Rubin): Let $\mathcal{M} = (M, <, P_i)_{i \in A}$ for some $A \subseteq \omega$. where P_i's are arbitrary unary predicates and where $<$ defines a linear order on infinite M. Then $\mathrm{Th}(\mathcal{M})$ is binary.

Question: If Th$(\omega, <, \dots)$ is small is it binary?

Theorem:

(1) There is no proper expansion of $(\omega, <)$ (or $(\omega + \omega^*, <)$) satisfying $CB(x = x) = \deg(x = x) = 1$.

(2) Any expansion of $(\omega, <)$ (or $(\omega + \omega^*, <)$) satisfying $CB(x = x) = 1$ and $\deg(x = x) > 1$ is an essentially unary expansion: Particularly

   (a) Expansion of $(\omega, <)$ satisfying $CB(x = x) = 1$ and $\deg(x = x) = d$ is definitionally equivalent to $(\omega, <, P_d)$.

   (b) Any expansion of $(\omega + \omega^*, <)$ in which ω is not a definable subset and satisfying $CB(x = x) = 1$ and $\deg(x = x) = d$ is definitionally equivalent to is a definitional expansion of $(\omega + \omega^*, <, B_{d,l})$ for some integer l.

Question 2: Is every expansions of $(\omega, <)$ with small theory essentially unary?

Answer to Q2: **No!**

There is small expansion of $(\omega, <)$ which is not essentially unary!

Example:

Let $\mathcal{M} = (\omega, <, P, f)$ where $P(x) \overset{def}{\Longleftrightarrow} x \in \{y^2 \mid y \in \omega\}$ and $f(x) = ([\sqrt{x}] + 1)^2 + x - [\sqrt{x}]^2$.
$\mathrm{CB}(x = x) = 2$.

$\mathcal{M}$ is not essentially unary expansion of $(\omega, <)$.