Lelek fan and generalizations of finite Gowers' FIN_k Theorem

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$\phi : X \longrightarrow Y$ is an epimorphism if

- ϕ - continuous
- ϕ - surjective homomorphism
- $(y_1, \ldots, y_n) \in R_j^Y \longrightarrow \exists (x_1, \ldots, x_n) \in R_j^X \ \phi(x_i) = y_i$

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Projective Fraïssé theory

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JPP $\forall A, B \in \mathcal{F}\ \exists C \in \mathcal{F}$ and epi $C \longrightarrow A$ and $C \longrightarrow B$

AP $\forall A, B, C \in \mathcal{F}$ and epi $f : B \longrightarrow A$ and $C \longrightarrow A\ \exists\ D \in \mathcal{F}$ and epi $k : D \longrightarrow B$ and $l : D \longrightarrow C$ such that $f \circ k = g \circ l$

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H $\forall A \in \mathcal{F}$ and epi $\phi_1, \phi_2 : \mathbb{F} \longrightarrow A\ \exists$ iso $\psi : \mathbb{F} \longrightarrow \mathbb{F}$ such that $\phi_2 = \phi_1 \circ \psi$

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Every projective Fraïssé class has a projective Fraïssé limit

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Theorem (Irwin, Solecki)

Every projective Fraïssé class has a projective Fraïssé limit which is unique up to an isomorphism.

Finite trees

$a, b \in (T, <_T)$ - a finite tree

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$$(a, b) \in R^T \longleftrightarrow (a = b \text{ or b } <_\text{T} \text{ a \& } \nexists \text{c} \in \text{T b } <_\text{T} \text{ c } <_\text{T} \text{ a})$$

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- $\mathcal{F}_<$ - finite fans with linearly ordered branches

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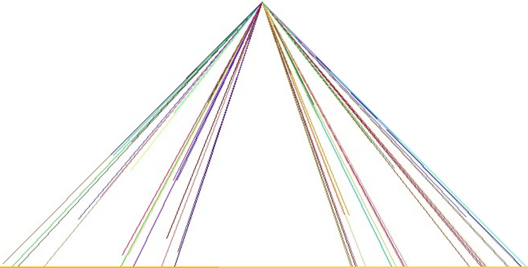

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$$\begin{aligned} h &\mapsto h^* \\ \pi \circ h &= h^* \circ \pi. \end{aligned}$$

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- does not contain any open subgroup, in particular it is not non-archimedean.
- is not locally compact.
- is (algebraically) simple.

Universal minimal flow

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Theorem

$M(G)$ exists and it is unique up to an isomorphism.

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Theorem

Let $\mathcal{K}$ be a projective Fraïssé class with a limit $\mathbb{K}$. If $\mathcal{K}$ satisfies the Ramsey property, then $\mathrm{Aut}(\mathbb{K})$ is extremely amenable.

Theorem (Ramsey)

For every $k \leq m$ and $r \geq 2$, there exists n such that for every colouring of k-element subsets of n with r-many colours there is a subset X of n of size m such that all k-element subsets of X have the same colour.

Ramsey and Dual Ramsey Theorem

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Theorem (Graham and Rothschild)

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Definition

$\mathcal{F}_<$ satisfies the Ramsey property if for every $A, B \in \mathcal{F}_<$ there exists $C \in \mathcal{F}_<$ such that for every colouring

$$c : \{C \longrightarrow A\} \longrightarrow \{1, 2, \ldots, r\}$$

there exists $f : C \longrightarrow B$ such that $\{B \longrightarrow A\} \circ f$ is monochromatic.

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Theorem (Hindman)

Let $c : \text{FIN}(\mathbb{N}) \longrightarrow \{1, 2, \ldots, r\}$ be a finite colouring. Then there is an infinite $A \subset \text{FIN}(\mathbb{N})$ such that FU(A) is monochromatic.

Operations on FIN_k

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Partial addition

$$\mathrm{supp}(p) \cap \mathrm{supp}(q) = \emptyset \ \longrightarrow \ p + q(n) = \max\{p(n), q(n)\}$$

Gowers' for FIN_k

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Theorem (Gowers)

Let $c : \mathrm{FIN}_k \longrightarrow \{1, 2, \ldots, r\}$ be a finite colouring. Then there is an infinite block sequence $B \subset \mathrm{FIN}_k$ such that $\langle B \rangle$ is monochromatic.

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$$T_{\vec{i}}(p) = T_1 \circ \ldots \circ T_k(p).$$

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Gowers with multiple operations

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Theorem

For every m, k, r, there exists n such that for every colouring $c : \mathrm{FIN}_k(n) \longrightarrow \{1, 2, \ldots, r\}$ there is a block sequence B of lenght m in $\mathrm{FIN}_k(n)$ such that $\langle B \rangle$ is monochromatic.

Pyramids

Higher dimensional Hindman

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Theorem (Milliken-Taylor)

For every m, r, d, there exists a natural number n such that for every colouring $c : \mathrm{FIN}_1^{[d]}(n) \longrightarrow \{1, 2, \ldots, r\}$, there is a block sequence B of length m such that $\langle B \rangle^{[d]}$ is monochromatic.

Pyramids

Theorem (Tyros)

For every triple m, k, r of positive integers, there exists n such that for every colouring $c : \mathrm{FIN}_k(n) \longrightarrow \{1, 2, \ldots, r\}$, there is a block sequence A of length m in $\mathrm{FIN}_1(n)$ such that any two elements in $\mathrm{FIN}_k(A)$ of the same type have the same colour.

Pyramids

Theorem (Tyros)

For every triple m, k, r of positive integers, there exists n such that for every colouring $c : \mathrm{FIN}_k(n) \longrightarrow \{1, 2, \ldots, r\}$, there is a block sequence A of length m in $\mathrm{FIN}_1(n)$ such that any two elements in $\mathrm{FIN}_k(A)$ of the same type have the same colour.

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$$T_{\vec{i}}(b)(\min \mathrm{supp}(T_{\vec{i}}(b))) = 1 = T_{\vec{i}}(b)(\max \mathrm{supp}(T_{\vec{i}}(b)))$$

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- lift B' to $B \subset \langle C \rangle$

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- "We can find a monochromatic subsequence in $\langle C \rangle$."

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$\mathrm{FIN}_{\mathrm{k,l}}$

Let k, m, r and $l \geq k$ be natural numbers. Then there exists a natural number n such that whenever we have a colouring $c : \mathrm{FIN}_k(n) \longrightarrow \{1, 2, \ldots, r\}$, there is a block sequence B in $\mathrm{FIN}_l(n)$ of length m such that the partial semigroup

$$\left\langle \bigcup_{\vec{i} \in P^l_{k+1}} T_{\vec{i}}(B) \right\rangle$$

is monochromatic.

Higher dimensions

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Theorem

Let (d, k, m, r) be a tuple of natural numbers. There exists n such that for every colouring $c : \mathrm{FIN}_k^{[d]}(n) \longrightarrow \{0, 1, \ldots, r\}$, there exists a block sequence B of length m such that $\langle B \rangle^{[d]}$ is monochromatic.

THANK YOU FOR YOUR ATTENTION!