Absoluteness via Resurrection

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Hilbert's Program (finding a complete and consistent theory for mathematics) had an abrupt stop after Gödel's Incompleteness Theorem in 1931.

The question whether it would be possible to have **empiric completeness** was left open, even if early results showed that ZFC does not have such a behavior.

Many of this results are obtained using forcing, thus in order to achieve empiric completeness we need to "rule it out". How?

Definition

A theory T has *generic absoluteness* for a family Θ of first-order formulas and a definable class Γ of CBAs iff in all models of T the truth values of formulas in Θ cannot be changed in forcing extensions obtained by CBAs in Γ which preserves T.

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We shall base on boolean valued models approach to forcing, and consider the following classes Γ of CBAs defined by properties interesting for forcing:

- Ω, the class of all CBAs,
- κ-distributive, κ-cc
- axiom-A, proper, semiproper (SP),
- stationary set preserving (SSP).

We shall equip a class Γ with two partial orders:

- $\mathbb{B} \leq_\Gamma \mathbb{C}$ iff there exists a complete homomorphism $i : \mathbb{C} \to \mathbb{B}$ such that the quotient algebra $\mathbb{B}/_{i[\dot{G}_\mathbb{C}]}$ is in Γ with boolean value $\mathbb{1}_\mathbb{C}$,
- $\mathbb{B} \leq_\Gamma^* \mathbb{C}$ iff there exists a complete *injective* homomorphism as above.

We denote by $\mathbb{U}_\kappa^\Gamma$ *(category forcing)* the set $\Gamma \cap H_\kappa$ ordered by $\leq_\Gamma$.

We say that Γ is iterable iff it is closed under two-step iterations, lottery sums and the order $\leq_\Gamma^*$ is closed for set-sized descending sequences of elements of Γ. Most of the interesting classes are iterable, with the notable exception of SSP.

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Fundamental generic absoluteness results are known in the literature for ZFC with large cardinals, e.g:

- ZFC: generic absoluteness for $\Gamma = \Omega$ and $\Theta = \Sigma^1_2(\mathbb{R})$ *(Shönfield)*
- ZFC $+\exists$class many Woodin cardinals limit of Woodin cardinals: generic absoluteness for $\Gamma = \Omega$ and Θ the formulas with real parameters relativized to $L(\mathbb{R})$ *(Woodin)*

More generic absoluteness results can be obtained in extensions of ZFC with forcing axioms.

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Intuitively, forcing axioms postulate that V is closed under taking suitable forcing extensions over small set models of ZFC.

Definition

FA(Γ) states that for all $\mathbb{B} \in \Gamma$ and collection $\mathcal{D}$ of $\aleph_1$-many dense subsets of $\mathbb{B}$, there exists a filter F meeting all of them.

Note that the same sentence for $\aleph_0$-many dense subsets is Baire's Category Theorem.

Other variations we will consider are BFA(Γ) (weakening) and FA^{++}(Γ) (strengthening). Recall that MM, PFA are shorthands for FA(SSP), FA(proper).

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- BFA(Γ) holds iff the class $\left\{\mathbb{B} \in \Gamma : H_{\aleph_2} \prec_1 V^{\mathbb{B}}\right\}$ is dense in $(\Gamma, \leq_{all})$,
- FA(SSP) holds iff $\{\mathbb{B} \in \mathrm{SSP} : \mathbb{B}$ is presaturated$\}$ is dense in $(\mathrm{SSP}, \leq_{all})$ *(class many Woodin cardinals)*,
- $\mathrm{FA}^{++}(\mathrm{SSP})$ holds iff the same class is dense in $(\mathrm{SSP}, \leq_{\mathrm{SSP}})$ *(class many Woodin cardinals)*,
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Examples of generic absoluteness results known in literature for extensions of ZFC with forcing axioms are:

- BFA(Γ) is equivalent to ZFC having generic absoluteness for Θ the Σ_1 formulas with parameters relativized to $H_{\aleph_2}$ and CBAs in Γ *(Bagaria)*,
- ZFC + MM^{+++} +∃class many superhuge cardinals has generic absoluteness for Γ = SSP and Θ the formulas relativized to $L([\mathrm{ON}]^{\aleph_1})$ *(Viale)*.

We show that strong generic absoluteness results can be obtained from resurrection axioms (of lower consistency strength).

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We show that strong generic absoluteness results can be obtained from resurrection axioms (of lower consistency strength).

We can develop the definition of the resurrection axiom starting from a model-theoretic point of view.

Theorem

Let $M \subset N$ be models of a language $\mathcal{L}$. Then TFAE:

- *M is existentially closed in N ($M \prec_1 N$),*
- *M has **resurrection**, i.e. it exists a larger $M' \supseteq N$ such that $M \prec M'$*

If we restrict the above properties to models of set theory of the form $H_{\mathfrak{c}}^M$ where $\mathfrak{c} = \aleph_2$ and consider only model extensions obtained by forcing in a fixed class Γ, we obtain respectively:

- M satisfies BFA(Γ),
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Resurrection axioms have been introduced recently by Hamkins and Johnstone, and are interesting since they can prove some consequences of FA, while having much lower consistency strength (for $\Gamma \neq \mathrm{SSP}$).

In particular, we have that:

- $\mathrm{RA}(\Gamma)$ for all mentioned Γ implies that $\mathfrak{c} \leq \aleph_2$,
- $\mathrm{RA}(\Gamma) + \neg\,\mathrm{CH}$ implies $\mathrm{BFA}(\Gamma)$,
- $\mathrm{FA}(\Gamma)$ is consistent relative to a supercompact cardinal *(Foreman, Magidor, Shelah)*,
- $\mathrm{RA}(\Gamma)$ for iterable Γ is consistent relative to a Mahlo cardinal *(Hamkins, Johnstone)*,
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The resurrection axiom is conveniently stated as a density property:

Definition

RA(Γ) holds iff the class $\left\{\mathbb{B} \in \Gamma : \; H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{V^{\mathbb{B}}}\right\}$ is dense in $(\Gamma, \leq_\Gamma)$.

From RA(Γ) we can already prove a weak form of generic absoluteness:

Theorem (Viale)

ZFC + RA(Γ) *has generic absoluteness for* Θ *the* Σ_2 *formulas with parameters relativized to* $H_{\mathfrak{c}}$ *and forcing in* Γ.

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Definition (iterated resurrection axiom)

$\mathrm{RA}_\omega(\Gamma)$ postulates that is possible to resurrect the theory of $H_\mathfrak{c}$ any fixed finite number of times.

Precisely, $\mathrm{RA}_\alpha(\Gamma)$ is the assertion:

> $\forall \beta < \alpha$ and $\forall N \supseteq M$ obtained by forcing in Γ,
> $\exists M' \supseteq N$ a further extension by a forcing in Γ,
> such that $H_\mathfrak{c}^M \prec H_\mathfrak{c}^{M'}$ and M' satisfies $\mathrm{RA}_\beta(\Gamma)$.

$$\alpha > \beta > \gamma > \dots$$

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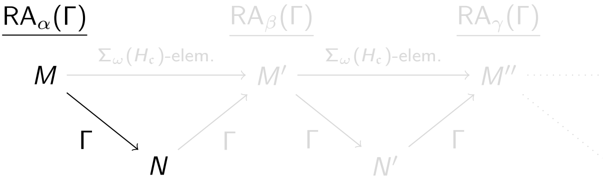

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Definition (iterated resurrection axiom)

$\mathsf{RA}_\omega(\Gamma)$ postulates that is possible to resurrect the theory of $H_\mathfrak{c}$ any fixed finite number of times.

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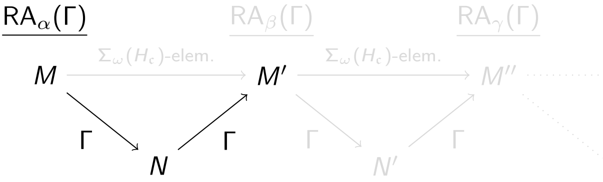

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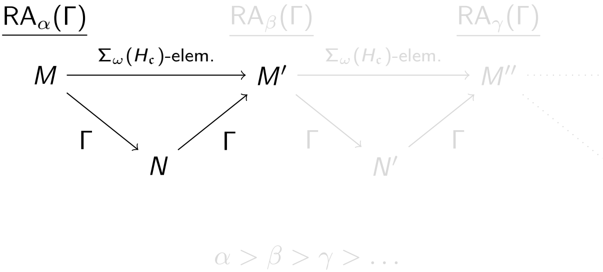

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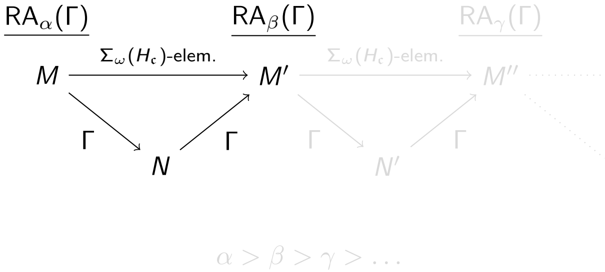

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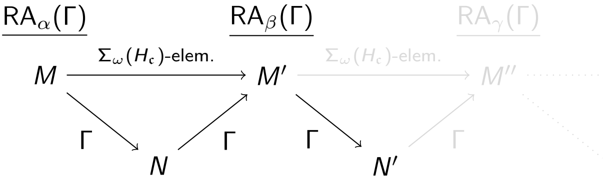

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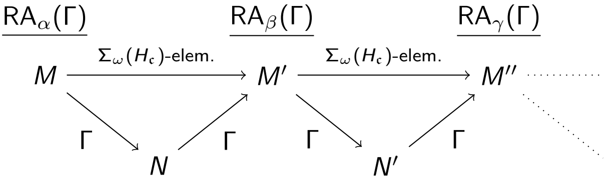

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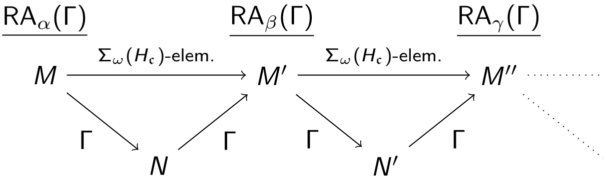

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Also the iterated resurrection axiom is conveniently stated as a density property:

Definition

$\mathsf{RA}_\alpha(\Gamma)$ holds iff for all $\beta < \alpha$ the class

$$\left\{\mathbb{B} \in \Gamma : \ H_{\mathfrak{c}} \prec H_{\mathfrak{c}}^{V^{\mathbb{B}}} \wedge V^{\mathbb{B}} \models \mathsf{RA}_\beta(\Gamma)\right\}$$

is dense in $(\Gamma, \leq_\Gamma)$.

From this strengthened axiom we can obtain:

Theorem (A., Viale)

$\mathsf{ZFC} + \mathsf{RA}_\omega(\Gamma)$ *has generic absoluteness for* Θ *the formulas relativized to* $H_{\mathfrak{c}}$ *and forcing in* Γ.

This directly improves the generic absoluteness result about $\mathsf{ZFC} + \mathsf{RA}(\Gamma)$, whereas with respect to Viale's absoluteness about $\mathsf{ZFC} + \mathsf{MM}^{+++} + \mathsf{LC}$:

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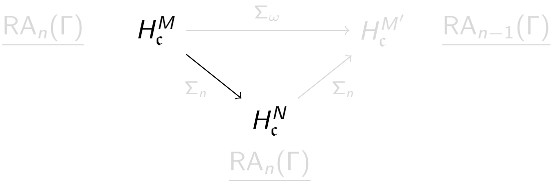

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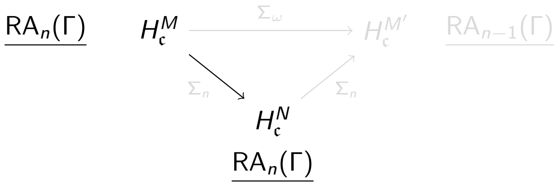

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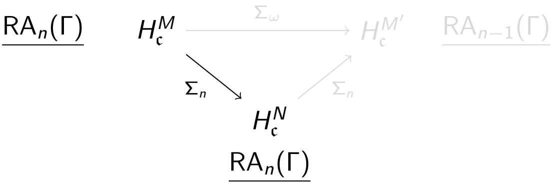

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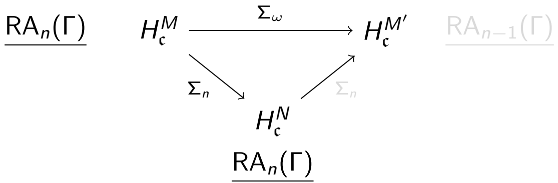

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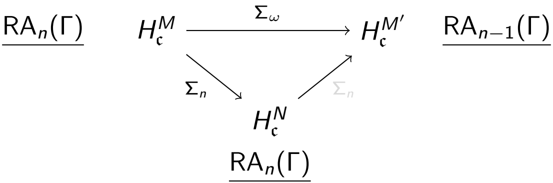

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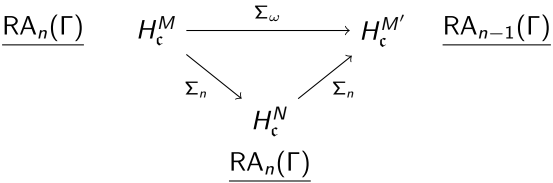

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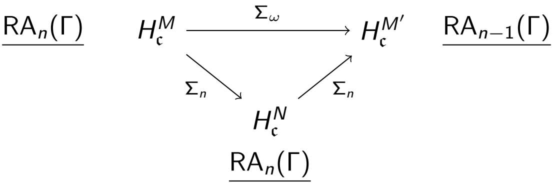

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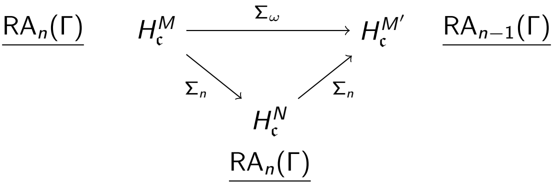

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The following holds:

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- $\mathsf{MM}^{+++} \Rightarrow \mathsf{RA}_{\mathsf{ON}}(\mathsf{SSP})$.

Sketchy proof.

To prove consistency of $\mathsf{RA}_\alpha(\Gamma)$ with Γ iterable (as for $\mathsf{FA}(\Gamma)$ and variations), we use lottery iteration forcing with respect to suitable fast-growing (Menas) function $f : \kappa \to \kappa$ for a large enough cardinal κ.

$$\begin{aligned} \mathbb{B}_0 &= 2 \\ \mathbb{B}_{\alpha+1} &= \mathbb{B}_\alpha * \dot{\mathbb{C}}_\alpha \text{ where } \dot{\mathbb{C}}_\alpha = \prod \left(\Gamma \cap H_{f(\alpha)}\right) a \\ \mathbb{B}_\alpha & \text{ for } \alpha \text{ limit is a lower bound in } \Gamma \text{ for the chain } \langle \mathbb{B}_\beta : \beta < \alpha \rangle \end{aligned}$$

For $\Gamma = \mathsf{SSP}$ we use the category forcing $\mathbb{U}_\kappa^{\mathsf{SSP}}$ for a large enough cardinal κ. □

Thanks for your attention!

Bibliography I

G. Audrito and M. Viale.
Absoluteness via resurrection.
arXiv:1404.2111, 2014.

J. D. Hamkins and T. A. Johnstone.
Resurrection axioms and uplifting cardinals.
arXiv:1307.3602, 2013.

K. Tsaprounis.
On resurrection axioms.
in preparation, 2013.

M. Viale.
Category forcings, MM^{+++}, and generic absoluteness for the theory of strong forcing axioms.
arXiv:1305.2058, 2013.