The similarity of positive Jonsson theories in admissible enrichments of signatures.

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Agreements about positivness and Δ

Let L is the language of the first order.

At is a set of atomic formulas of this language.

$B^+(At)$ is containing all the atomic formulas, and closed under positive Boolean combination and for sub-formulas and substitution of variables.

$L^+ = Q(B^+(At)$ is the set of formulas in prenex normal type obtained by application of quantifiers ($\forall$ and $\exists$) to $B^+(At)$.

$B(L^+)$ is any Boolean combination of formulas from L^+

$\Delta \subseteq B(L^+)$

Δ-Homomorphism

Let M and N are the structure of language, $\Delta \subseteq B(L^+)$.
The map $h : M \to N$ Δ-homomorphism (symbolically $h : M \leftrightarrow_\Delta N$, if for any $\varphi(\bar{x}) \in \Delta, \forall \bar{a} \in M$ such that $M \models \varphi(\bar{a})$, we have that $N \models \varphi(h(\bar{a}))$.

The model M is said to begin in N and we say that M continues to N, with $h(M)$ is a continuation of M. If the map h is injective, we say that h immersion M into N (symbolically $h : M \longleftrightarrow_{\Delta} N$). In the following we will use the terms Δ-continuation and Δ-immersion.

Δ-Joint Embedding Property and Δ-Amalgamation Property

Definition

The theory T admits $\Delta - JEP$, if for any $A, B \in ModT$ there are exist $C \in ModT$ and Δ-homomorphism's $h_1 : A \to_\Delta C$, $h_2 : B \to_\Delta C$.

Definition

The theory T admits $\Delta - AP$, if for any $A, B, C \in ModT$ with $h_1 : A \to_\Delta C$, $g_1 : A \to_\Delta B$, where h_1, g_1 are Δ-homomorphism's, there are exist $D \in Modt$ and $h_2 : C \to_\Delta D$, $g_2 : B \to_\Delta D$ where h_2, g_2 are Δ-homomorphism's such that $h_2 \circ h_1 = g_2 \circ g_1$.

Definition

The theory T is called Δ-positive Jonsson ($\Delta - PJ$) theory if it satisfies the following conditions:

1. T has an infinite model;
2. T positive $\forall\exists$-axiomatizable;
3. T admits $\Delta - JEP$;
4. T admits $\Delta - AP$.

When $\Delta = B(At)$ and we shall consider only Δ-immersions, we obtain the usual Jonsson theory, the difference only that it has only positive $\forall\exists$-axiom.

The lattice of positive formulas

Let $\varphi, \psi \in PE_n(T)$ and $\varphi \cap \psi = 0$, where 0 - 0 lattice $PE_n(T)$. Then ψ is called the complement of φ, if $\varphi \cup \psi = 1$, where 1 - 1 of the lattice $PE_n(T)$; ψ is a weak complement of φ, if for all $\mu \in PE_n(T)$ $(\varphi \cup \psi) \cap \mu = 0 \Rightarrow \mu = 0$. φ is called weakly complemented, if φ has a weak complement. $PE_n(T)$ is called weakly complemented if every $\varphi \in PE_n(T)$ is weakly complemented.

Theorem 1. [AY]

Let T - complete for $\exists$-sentences Jonsson's theory. Then the following conditions are equivalent:

1. T is perfect;
2. T^* is model companion of T;

Totaly categorical Jonsson universal

Well known the following question : Is there exist ω-categorical but non ω_1-categorical universal?
If we have proposed the negative answer , then in the frame of such Jonsson universal we can obtain that the center of this one is not finite-axiomatizable.

Theorem 2. [HML]

1. The theory T is model complete if and only if every formula is persist with respect to the submodels $ModT$.
2. The theory T is model complete if and only if every formula is persist under extensions of models in $ModT$.

Theorem 3. [W]

Theory T positively model-complete if and only if each $\varphi^T \in E_n(T)$ has a positive existential complement.

Relation between completeness and model-completeness

Theorem. [Lin]

Let T - a complete inductive theory and κ-categorical, where $\kappa \geq$ card(L). Then theory T is model complete.

Theorem 4. [AY]

Let T - perfect Jonsson theory. The following conditions are equivalent:

1. T - is complete;
2. T - is model complete.

Positively existentially closeness

Definition

The model A of the theory T is Δ-positively existentially closed, if for any Δ-immersion $h : A \to_\Delta B$ and any $\bar{a} \in A$ and $\varphi(\bar{x}, \bar{y}) \in \Delta$, $B \models \exists \bar{y} \varphi(h(\bar{a}), \bar{y}) \Rightarrow A \models \exists \bar{y} \varphi(\bar{a}, \bar{y})$.

Theorem 5. [W]

Existential formulas φ is invariant in $Mod(Th_{\forall\exists}(E_T))$, where $E(T)$ - the class of existentially closed models of T, if and only if φ^T is weakly complemented in $E(T)$.

Model companion and model completion

Theorem 6. [HML]

1. Let T' - a model companion of the theory T, where T - the universal theory. In this case, T' - a model completion of T, if and only if the theory T admits elimination of quantifiers.
2. Let T' - a model companion of T. In this case, T' - a model completion of T, if and only if the theory T has the amalgamation property.

Theorem 7. [W]

The theory T has a model completion if and only if $E_n(T)$ - algebra of Stone.

Theorem 8. [W]

The theory T has a model completion if and only if each $\varphi^T \in E_n(T)$ has a weak quantifier-free complement.

Theorem 9. [AY]

Let T - complete for the Σ^{+}-sentences $\Delta - PJ$ theory, T^{*}_{Δ} - center of the theory of T. Then

1. T^{*}_{Δ} admits elimination of quantifiers if and only if every $\varphi \in PE_n(T)$ is quantifier-free complement;
2. T^{*}_{Δ} $\Delta - PJ$-positively model-complete if and only if every $\varphi \in PE_n(T)$ is a positive existential complement.

$\Delta - PJ$-perfectness

Theorem 10. [AY]

Let $T - \Delta - PJ$-theory. Then the following conditions are equivalent:

1. $T - \Delta - PJ$-perfect;
2. $PE_n(T)$ is complemented weak;
3. $PE_n(T)$ - Stone lattice.

Theorem 11. [AY]

Let $T - \Delta - PJ$-theory. Then the following conditions are equivalent:

1. $T^*_\Delta - \Delta - PJ$-theory;
2. each $\varphi \in PE_n(T)$ has a weak quantifier-free complement.

Let T is an arbitrary $\Delta - PJ$ theory in first order signature σ. Let C is a semantic model of T. $A \subseteq C$. Let $\sigma_\Gamma(A) = \sigma \cup \{c_a | a \in A\} \cup \Gamma$, where $\Gamma = \{P\} \cup \{c\}$. Let consider the following theory $T_\Gamma^{PJ}(A) = Th_{\forall\exists^+}(C, a)_{a\in A} \cup \{P(c_a | a \in A)\} \cup \{P(c)\} \cup \{"P \subseteq "\}$, where $\{"P \subseteq "\}$, is infinite set of the sentences, expressing fact, that the interpretation of the symbol P is existentially closed submodel in the signature σ. The requirement of existential closeness for a submodel is essential in that sense, that it should not be finite . This theory is not necessary complete.

Let consider all completions of T^* for T in σ_Γ, where $\Gamma = \{P\} \cup \{c\}$. Due to that T^* is $\Delta - PJ$-theory, it has its center and we call it as T^C. By a restriction of T^C till a signature $\sigma \cup \{P\}$ theory T^C became complete type. This type we call as central type of theory T. It will be noted that all semantic models are elementarily equivalent between each other.

Stability

Let S_{Γ}^{PJ} is the set of all $\exists^{+}$-completions of a theory $T_{\Gamma}^{PJ}(A)$. Let λ is an arbitrary cardinal. $\Delta - PJ$ theory is J-P-λ-stable, if $\left|S_{\Gamma}^{PJ}\right| \leq \lambda$ for any A, such that $|A| \leq \lambda$.

Theorem. [AY]

Let T be a $\exists^{+}$-complete perfect Δ-PJ theory. Then the following conditions equivalent:

1. theory T^{C} is P-λ-stable
2. theory T^{*} is J-P-λ-stable.

Syntactical similarity

Let $F_n(T)$, $n < \omega$, be the of formulas of T with exactly n free variables $v_1, \ldots, v_n$ and $F(T) = \bigcup_n F_n(T)$.

Definition

Complete theories T_1 and T_2 are syntactically similar if and only if there exists a bijection $f : F(T_1) \to F(T_2)$ such that:

- $f \restriction F_n(T_1)$ is an isomorphism of the Boolean algebras $F(T_1)$ and $F(T_2)$, $n < \omega$;
- $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in F_{n+1}(T)$, $n < \omega$;
- $f(v_1 = v_2) = (v_1 = v_2)$.

Let T is arbitrary $\Delta - PJ$-theory, then $E^+(T) = \bigcup_{n<\omega} E_n^+(T)$, where $E_n^+(T)$ - is the lattice of positive existential formulas with exactly n free variables.

Let T_1 and T_2 - $\Delta - PJ$-theories.

We shall say that, T_1 и T_2 - $\Delta - PJ$-syntactically similar, if and only if there exist a bijection $f : E^+(T_1) \to E^+(T_2)$ such that

1. the restriction of f up $E_n^+(T_1)$ is isomorphism of the $E_n^+(T_1)$ and $E_n^+(T_2)$, $n < \omega$;
2. $f(\exists v_{n+1}\varphi) = \exists v_{n+1} f(\varphi)$, $\varphi \in E_n^+(T)$, $n < \omega$,
3. $f(v_1 = v_2) = (v_1 = v_2)$.

Let us recall the definition of a polygon .

Definition

By a polygon over a monoid S we mean a structure with only unary functions $\langle A; f_{\alpha:\alpha\in S}\rangle$ such that
(i) $f_e(a) = a, \forall a \in A$, where e is the unit of S;
(ii)$f_{\alpha\beta}(a) = f_\alpha(f_\beta(a)), \forall \alpha, \beta \in S, \forall a \in A$.

Theorem 12. [AY]

Let T_1 and T_2 be $\exists^+$-complete perfect $\Delta - PJ$ theories. Then the following conditions equivalent:

1. T_1^* and T_2^* are $\Delta - PJ$-syntactically similar;
2. T_1^C and T_2^C are syntactical similar as complete theories.

Corollary 1. [AY]

For each Σ-complete perfect Jonsson $\Delta - PJ$ theory there exist $\Delta - PJ$ syntactically similar a Σ-complete perfect Jonsson $\Delta - PJ$ theory of polygons, such that its center is model complete.

(1) By a pure triple we mean $\langle A, \Gamma, M \rangle$, where A is not empty set, Γ is a permutation group on A, and M is a family of subsets of A such that $M \in M \Rightarrow g(M) \in M$ for every $g \in \Gamma$.

(2) If $\langle A_1, \Gamma_1, M_1 \rangle$ and $\langle A_2, \Gamma_2, M_2 \rangle$ are pure triples, and $\psi : A_1 \to A_2$ is a bijection, then ψ is an isomorphism, if

(i) $\Gamma_2 = \{\psi g \psi^{-1} : g \in \Gamma_1\}$;

(ii) $M_2 = \{\psi(E) : E \in M_1\}$.

Definition

The pure triple $\langle |C|, G, N \rangle$ is called the semantically triple of T (abbreviated s.t.), where $|C|$ is the universe of $C, G = Aut(C)$ and N is the class of all subsets of $|C|$ which are universes of suitable elementary submodels of C.

Complete theories T_1 and T_2 are semantically similar is and only if their semantic triples are isomorphic.

The relation between both types of similarities

Definition

A property (or a notion) of theories (or models, or elements of models) is called semantic if and if it is invariant relative to seemantic similarity.

Proposition 1. [TGM]

If T_1 and T_2 are syntactically similar, then and are semantically similar. The converse implication fails.

Proposition 2. [TGM]

The following properties and notions are semantic:

(1) type;

(2) forking;

(3) λ-stability;

(4) Lascar rank;

(5) Strong type;

(6) Morley sequence;

(7) Orthogonality, regularity of types;

(8) $I(\aleph_\alpha, T)$ - the spectrum function.

By virtue of this notice we can say that all above mentioned properties and notions from **Proposition 2** in the class of centers of $\exists^+$-complete perfect $\Delta - PJ$ theories are semantic. Moreover if we are consider above mentioned enrichments of signatures of such theories and we will consider central types of ones we got that the situation will not change. And finally it is appropriate to consider the $\Delta - PJ$-analogues of the list of semantic properties and notions from classical model theory.

Thanks a lot for your attention!