Free idempotent generated semigroups over bands

Dandan Yang

joint work with Vicky Gould

Novi Sad, June 2013

String rewriting systems

Let Σ be a finite alphabet and Σ^* the set of all words on Σ.

A **string-rewriting system** R on Σ is a subset of $\Sigma^* \times \Sigma^*$.

String rewriting systems

Let Σ be a finite alphabet and Σ^* the set of all words on Σ.

A **string-rewriting system** R on Σ is a subset of $\Sigma^* \times \Sigma^*$.

A **single-step reduction relation** on Σ^* is defined by

$$u \longrightarrow v \Longleftrightarrow (\exists (l, r) \in R)\ (\exists\ x, y \in \Sigma^*)\ u = xly \text{ and } v = xry.$$

String rewriting systems

Let Σ be a finite alphabet and Σ^* the set of all words on Σ.

A **string-rewriting system** R on Σ is a subset of $\Sigma^* \times \Sigma^*$.

A **single-step reduction relation** on Σ^* is defined by

$$u \longrightarrow v \Longleftrightarrow (\exists (l, r) \in R) \ (\exists \ x, y \in \Sigma^*) \ u = xly \text{ and } v = xry.$$

The **reduction relation** on Σ^* induced by R is the reflexive, transitive closure of $\longrightarrow$ and is denoted by $\stackrel{*}{\longrightarrow}$.

String rewriting systems

Let Σ be a finite alphabet and Σ^* the set of all words on Σ.

A **string-rewriting system** R on Σ is a subset of $\Sigma^* \times \Sigma^*$.

A **single-step reduction relation** on Σ^* is defined by

$$u \longrightarrow v \Longleftrightarrow (\exists (l, r) \in R)\ (\exists\ x, y \in \Sigma^*)\ u = xly \text{ and } v = xry.$$

The **reduction relation** on Σ^* induced by R is the reflexive, transitive closure of $\longrightarrow$ and is denoted by $\stackrel{*}{\longrightarrow}$.

The structure $S = (\Sigma^*, R)$ is called a **reduction system**.

String rewriting systems

A word in Σ^* is in **normal form** if we cannot apply a relation in R.

String rewriting systems

A word in Σ^* is in **normal form** if we cannot apply a relation in R.

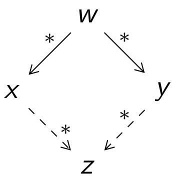
S is **noetherian** if $\nexists\ w = w_0 \longrightarrow w_1 \longrightarrow w_2 \longrightarrow \cdots$

String rewriting systems

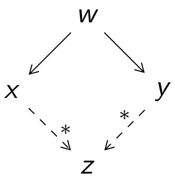
A word in Σ^* is in **normal form** if we cannot apply a relation in R.

S is **noetherian** if $\nexists\ w = w_0 \longrightarrow w_1 \longrightarrow w_2 \longrightarrow \cdots$

S is confluent



S is locally confluent

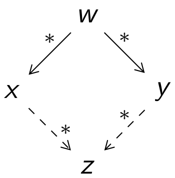


String rewriting systems

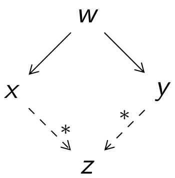
A word in Σ^* is in **normal form** if we cannot apply a relation in R.

S is **noetherian** if $\nexists\ w = w_0 \longrightarrow w_1 \longrightarrow w_2 \longrightarrow \cdots$

S is confluent



S is locally confluent



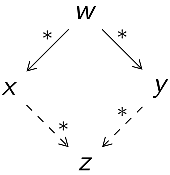
Facts:

String rewriting systems

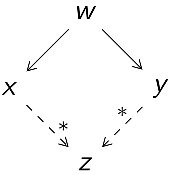
A word in Σ^* is in **normal form** if we cannot apply a relation in R.

S is **noetherian** if $\nexists\ w = w_0 \longrightarrow w_1 \longrightarrow w_2 \longrightarrow \cdots$

S is confluent



S is locally confluent



Facts:

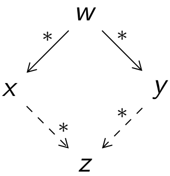
1. let ρ be the congruence generated by R. Then S is noetherian and confluent implies **every ρ-class contains a unique normal form.**

String rewriting systems

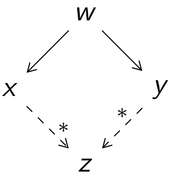
A word in Σ^* is in **normal form** if we cannot apply a relation in R.

S is **noetherian** if $\nexists\ w = w_0 \longrightarrow w_1 \longrightarrow w_2 \longrightarrow \cdots$

S is confluent



S is locally confluent



Facts:

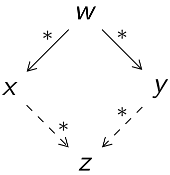
1. let ρ be the congruence generated by R. Then S is noetherian and confluent implies **every ρ-class contains a unique normal form.**

String rewriting systems

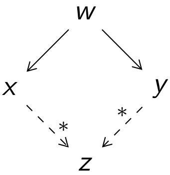
A word in Σ^* is in **normal form** if we cannot apply a relation in R.

S is **noetherian** if $\nexists\ w = w_0 \longrightarrow w_1 \longrightarrow w_2 \longrightarrow \cdots$

S is confluent



S is locally confluent



Facts:

1. let ρ be the congruence generated by R. Then S is noetherian and confluent implies **every ρ-class contains a unique normal form.**
2. If S is noetherian, then

$$\textbf{confluent} \iff \textbf{locally confluent}.$$

Free idempotent generated semigroups

Let S be a semigroup with E a set of all idempotents of S.

For any $e, f \in E$, define

$$e \leq_{\mathcal{R}} f \Leftrightarrow fe = e \text{ and } e \leq_{\mathcal{L}} f \Leftrightarrow ef = e.$$

Free idempotent generated semigroups

Let S be a semigroup with E a set of all idempotents of S.

For any $e, f \in E$, define

$$e \leq_{\mathcal{R}} f \Leftrightarrow fe = e \text{ and } e \leq_{\mathcal{L}} f \Leftrightarrow ef = e.$$

Note $e \leq_{\mathcal{R}} f$ $(e \leq_{\mathcal{L}} f)$ implies both ef and fe are idempotents.

Free idempotent generated semigroups

Let S be a semigroup with E a set of all idempotents of S.

For any $e, f \in E$, define

$$e \leq_{\mathcal{R}} f \Leftrightarrow fe = e \text{ and } e \leq_{\mathcal{L}} f \Leftrightarrow ef = e.$$

Note $e \leq_{\mathcal{R}} f$ ($e \leq_{\mathcal{L}} f$) implies both ef and fe are idempotents.

We say that (e, f) is **a basic pair** if

$$e \leq_{\mathcal{R}} f, f \leq_{\mathcal{R}} e, e \leq_{\mathcal{L}} f \text{ or } f \leq_{\mathcal{L}} e$$

i.e. $\{e, f\} \cap \{ef, fe\} \neq \emptyset$; then ef, fe are said to be **basic products.**

Free idempotent generated semigroups

Let S be a semigroup with E a set of all idempotents of S.

For any $e, f \in E$, define

$$e \leq_{\mathcal{R}} f \Leftrightarrow fe = e \text{ and } e \leq_{\mathcal{L}} f \Leftrightarrow ef = e.$$

Note $e \leq_{\mathcal{R}} f$ ($e \leq_{\mathcal{L}} f$) implies both ef and fe are idempotents.

We say that (e, f) is **a basic pair** if

$$e \leq_{\mathcal{R}} f, f \leq_{\mathcal{R}} e, e \leq_{\mathcal{L}} f \text{ or } f \leq_{\mathcal{L}} e$$

i.e. $\{e, f\} \cap \{ef, fe\} \neq \emptyset$; then ef, fe are said to be **basic products.**

Under basic products, E satisfies a number of axioms.

Free idempotent generated semigroups

Let S be a semigroup with E a set of all idempotents of S.

For any $e, f \in E$, define

$$e \leq_{\mathcal{R}} f \Leftrightarrow fe = e \text{ and } e \leq_{\mathcal{L}} f \Leftrightarrow ef = e.$$

Note $e \leq_{\mathcal{R}} f$ $(e \leq_{\mathcal{L}} f)$ implies both ef and fe are idempotents.

We say that (e, f) is **a basic pair** if

$$e \leq_{\mathcal{R}} f, f \leq_{\mathcal{R}} e, e \leq_{\mathcal{L}} f \text{ or } f \leq_{\mathcal{L}} e$$

i.e. $\{e, f\} \cap \{ef, fe\} \neq \emptyset$; then ef, fe are said to be **basic products.**

Under basic products, E satisfies a number of axioms.

A **biordered set** is a partial algebra satisfying these axioms.

Free idempotent generated semigroups

Let S be a semigroup with biordered set E.

Free idempotent generated semigroups

Let S be a semigroup with biordered set E.

The free idempotent generated semigroup $\mathrm{IG}(E)$ is defined by

$$\mathrm{IG}(E) = \langle \overline{E} : \bar{e}\bar{f} = \overline{ef},\ e, f \in E, \{e, f\} \cap \{ef, fe\} \neq \emptyset \rangle.$$

where $\overline{E} = \{\bar{e} : e \in E\}$.

Free idempotent generated semigroups

Let S be a semigroup with biordered set E.

The free idempotent generated semigroup IG(E) is defined by

$$\mathrm{IG}(E) = \langle \overline{E} : \bar{e}\bar{f} = \overline{ef},\ e, f \in E, \{e, f\} \cap \{ef, fe\} \neq \emptyset \rangle.$$

where $\overline{E} = \{\bar{e} : e \in E\}$.

Facts:

Free idempotent generated semigroups

Let S be a semigroup with biordered set E.

The free idempotent generated semigroup $\mathsf{IG}(E)$ is defined by

$$\mathsf{IG}(E) = \langle \overline{E} : \bar{e}\bar{f} = \overline{ef},\ e, f \in E, \{e, f\} \cap \{ef, fe\} \neq \emptyset \rangle.$$

where $\overline{E} = \{\bar{e} : e \in E\}$.

Facts:

1. $\phi : \mathsf{IG}(E) \to \langle E \rangle$, given by $\bar{e}\phi = e$ is an epimorphism.

Free idempotent generated semigroups

Let S be a semigroup with biordered set E.

The free idempotent generated semigroup $\mathrm{IG}(E)$ is defined by

$$\mathrm{IG}(E) = \langle \overline{E} : \bar{e}\bar{f} = \overline{ef},\ e, f \in E, \{e, f\} \cap \{ef, fe\} \neq \emptyset \rangle.$$

where $\overline{E} = \{\bar{e} : e \in E\}$.

Facts:

1. $\phi : \mathrm{IG}(E) \to \langle E \rangle$, given by $\bar{e}\phi = e$ is an epimorphism.
2. $E \cong E(\mathrm{IG}(E))$

Let S be a semigroup with biordered set E.

The free idempotent generated semigroup $\mathrm{IG}(E)$ is defined by

$$\mathrm{IG}(E) = \langle \overline{E} : \bar{e}\bar{f} = \overline{ef},\ e, f \in E, \{e, f\} \cap \{ef, fe\} \neq \emptyset \rangle.$$

where $\overline{E} = \{\bar{e} : e \in E\}$.

Facts:

1. $\phi : \mathrm{IG}(E) \to \langle E \rangle$, given by $\bar{e}\phi = e$ is an epimorphism.
2. $E \cong E(\mathrm{IG}(E))$

Note $\mathrm{IG}(E)$ naturally gives us a reduction system $(\overline{E}^*, R)$, where

$$R = \{(\bar{e}\bar{f}, \overline{ef}) :\ (e, f) \text{ is a basic pair}\}.$$

Let S be a semigroup with biordered set E.

The free idempotent generated semigroup $\mathrm{IG}(E)$ is defined by

$$\mathrm{IG}(E) = \langle \overline{E} : \bar{e}\bar{f} = \overline{ef},\ e, f \in E, \{e, f\} \cap \{ef, fe\} \neq \emptyset \rangle.$$

where $\overline{E} = \{\bar{e} : e \in E\}$.

Facts:

1. $\phi : \mathrm{IG}(E) \to \langle E \rangle$, given by $\bar{e}\phi = e$ is an epimorphism.
2. $E \cong E(\mathrm{IG}(E))$

Note $\mathrm{IG}(E)$ naturally gives us a reduction system $(\overline{E}^*, R)$, where

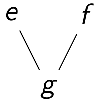
$$R = \{(\bar{e}\bar{f}, \overline{ef}) :\ (e, f) \text{ is a basic pair}\}.$$

Aim Today: To study the general structure of $\mathrm{IG}(E)$, for some bands.

IG(E) over semilattices

IG(E) is not necessarily regular.

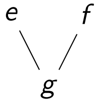
IG(E) is not necessarily regular. Consider a semilattice



Then $\overline{e}\ \overline{f} \in \mathrm{IG}(E)$ is not regular.

IG(E) over semilattices

IG(E) is not necessarily regular. Consider a semilattice



Then $\overline{e}\ \overline{f} \in$ IG(E) is not regular.

What other structures does IG(E) might have?

IG(E) over semilattices

Green's star equivalences

For any $a, b \in S$,

$$a \; \mathcal{L}^* \; b \iff (\forall x, y \in S^1) \; (ax = ay \Leftrightarrow bx = by).$$

Green's star equivalences

For any $a, b \in S$,

$$a \ \mathcal{L}^* \ b \iff (\forall x, y \in S^1) \ (ax = ay \Leftrightarrow bx = by).$$

Dually, the relation $\mathcal{R}^*$ is defined on S.

Green's star equivalences

For any $a, b \in S$,

$$a \ \mathcal{L}^* \ b \iff (\forall x, y \in S^1) \ (ax = ay \Leftrightarrow bx = by).$$

Dually, the relation $\mathcal{R}^*$ is defined on S.

Note $\mathcal{L} \subseteq \mathcal{L}^*$, $\mathcal{R} \subseteq \mathcal{R}^*$.

Green's star equivalences

For any $a, b \in S$,

$$a \ \mathcal{L}^* \ b \iff (\forall x, y \in S^1) \, (ax = ay \Leftrightarrow bx = by).$$

Dually, the relation $\mathcal{R}^*$ is defined on S.

Note $\mathcal{L} \subseteq \mathcal{L}^*$, $\mathcal{R} \subseteq \mathcal{R}^*$.

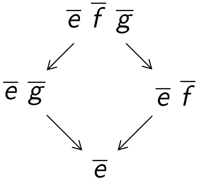
Definition A semigroup S is called **abundant** if each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent of S.

IG(E) over semilattices

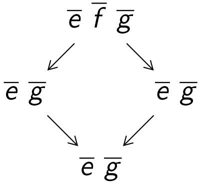
Lemma $(\overline{E}^*, R)$ is locally confluent.

Lemma $(\overline{E}^*, R)$ is locally confluent.

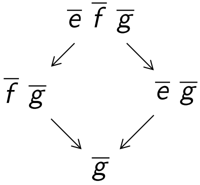
(i) $e \le f, f \le g$



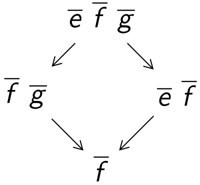
(ii) $e \le f, f \ge g$



(iii) $e \ge f, f \ge g$



(iv) $e \ge f, f \le g$



Lemma $(\overline{E}^*, R)$ is noetherian.

IG(E) over semilattices

Lemma $(\overline{E}^*, R)$ is noetherian.

Corollary Every element $\overline{x_1} \cdots \overline{x_n} \in \text{IG}(E)$ has a unique normal form.

IG(E) over semilattices

Lemma $(\overline{E}^*, R)$ is noetherian.

Corollary Every element $\overline{x_1} \cdots \overline{x_n} \in \mathrm{IG}(E)$ has a unique normal form.

Theorem IG(E) is abundant, and so adequate.

Lemma $(\overline{E}^*, R)$ is noetherian.

Corollary Every element $\overline{x_1} \cdots \overline{x_n} \in \mathrm{IG}(E)$ has a unique normal form.

Theorem IG(E) is abundant, and so adequate.

Note Adequate semigroups belong to a quasivariety of algebras introduced in **York** by **Fountain** over 30 years ago, for which the free objects have recently been described.

IG(E) over simple bands

Recall that a band E is a semilattice Y of rectangular bands E_α, where $\alpha \in Y$.

IG(E) over simple bands

Recall that a band E is a semilattice Y of rectangular bands E_α, where $\alpha \in Y$.

Definition A band E is called a **simple band** if each E_α is either a left zero band or a right zero band, for all $\alpha \in Y$.

IG(E) over simple bands

Recall that a band E is a semilattice Y of rectangular bands E_α, where $\alpha \in Y$.

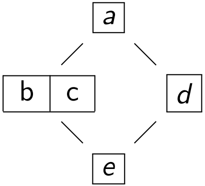
Definition A band E is called a **simple band** if each E_α is either a left zero band or a right zero band, for all $\alpha \in Y$.
Note We lose the uniqueness of normal form here!

IG(E) over simple bands

Recall that a band E is a semilattice Y of rectangular bands E_α, where $\alpha \in Y$.

Definition A band E is called a **simple band** if each E_α is either a left zero band or a right zero band, for all $\alpha \in Y$.

Note We lose the uniqueness of normal form here! Consider the following simple band

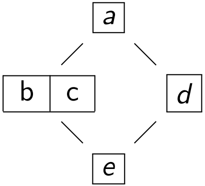


	a	b	c	d	e
a	a	b	c	d	e
b	b	b	c	e	e
c	b	b	c	e	e
d	d	e	e	d	e
e	e	e	e	e	e

IG(E) over simple bands

Recall that a band E is a semilattice Y of rectangular bands E_α, where $\alpha \in Y$.

Definition A band E is called a **simple band** if each E_α is either a left zero band or a right zero band, for all $\alpha \in Y$.
Note We lose the uniqueness of normal form here! Consider the following simple band



	a	b	c	d	e
a	a	b	c	d	e
b	b	b	c	e	e
c	b	b	c	e	e
d	d	e	e	d	e
e	e	e	e	e	e

Clearly, $\overline{c}\ \overline{d} = \overline{c}\ \overline{ad} = \overline{c}\ \overline{a}\ \overline{d} = \overline{ca}\ \overline{d} = \overline{b}\ \overline{d}$

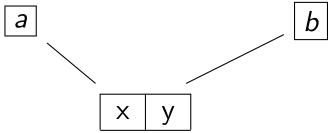
IG(E) over simple bands

Note IG(E) does not have to be abundant.

IG(E) over simple bands

Note IG(E) does not have to be abundant.

Let $E = \{a, b, x, y\}$ be a simple band with

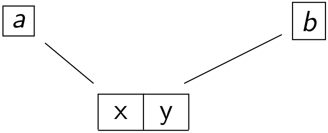


	a	b	x	y
a	a	y	x	y
b	y	b	x	y
x	x	y	x	y
y	y	y	x	y

IG(E) over simple bands

Note IG(E) does not have to be abundant.

Let $E = \{a, b, x, y\}$ be a simple band with



	a	b	x	y
a	a	y	x	y
b	y	b	x	y
x	x	y	x	y
y	y	y	x	y

Note $\overline{a}$ $\overline{b}$ does not $\mathcal{R}^*$-related to any element in E.

IG(E) over simple bands

Let $U \subseteq E(S)$. For any $a, b \in S$,

$$a \, \widetilde{\mathcal{L}}_U \, b \iff (\forall e \in U)\, (ae = a \Leftrightarrow be = b).$$
$$a \, \widetilde{\mathcal{R}}_U \, b \iff (\forall e \in U)\, (ea = a \Leftrightarrow eb = b).$$

IG(E) over simple bands

Let $U \subseteq E(S)$. For any $a, b \in S$,

$$a \ \widetilde{\mathcal{L}}_U \ b \iff (\forall e \in U)\ (ae = a \Leftrightarrow be = b).$$

$$a \ \widetilde{\mathcal{R}}_U \ b \iff (\forall e \in U)\ (ea = a \Leftrightarrow eb = b).$$

Note $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}_U$, $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_U$.

Let $U \subseteq E(S)$. For any $a, b \in S$,

$$a \, \widetilde{\mathcal{L}}_U \, b \iff (\forall e \in U) \, (ae = a \Leftrightarrow be = b).$$
$$a \, \widetilde{\mathcal{R}}_U \, b \iff (\forall e \in U) \, (ea = a \Leftrightarrow eb = b).$$

Note $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}_U$, $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_U$.

Definition A semigroup S with $U \subseteq E(S)$ is called **weakly U-abundant** if each $\widetilde{\mathcal{L}}_U$-class and each $\widetilde{\mathcal{R}}_U$-class contains an idempotent in U, and U is called the **distinguished set** of S.

Let $U \subseteq E(S)$. For any $a, b \in S$,

$$a \; \widetilde{\mathcal{L}}_U \; b \iff (\forall e \in U) \; (ae = a \Leftrightarrow be = b).$$
$$a \; \widetilde{\mathcal{R}}_U \; b \iff (\forall e \in U) \; (ea = a \Leftrightarrow eb = b).$$

Note $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}_U$, $\mathcal{R} \subseteq \mathcal{R}^* \subseteq \widetilde{\mathcal{R}}_U$.

Definition A semigroup S with $U \subseteq E(S)$ is called **weakly U-abundant** if each $\widetilde{\mathcal{L}}_U$-class and each $\widetilde{\mathcal{R}}_U$-class contains an idempotent in U, and U is called the **distinguished set** of S.

Definition A weakly U-abundant semigroup semigroup S satisfies the **congruence condition** if $\widetilde{\mathcal{L}}_U$ is a right congruence and $\widetilde{\mathcal{R}}_U$ is a left congruence.

IG(E) over simple bands

Lemma For any $e \in E_\alpha, f \in E_\beta$, (e, f) is basic pair in E if and only if (α, β) is a basic pair in Y.

Lemma For any $e \in E_\alpha, f \in E_\beta$, (e, f) is basic pair in E if and only if (α, β) is a basic pair in Y.

Lemma Let $\theta : S \longrightarrow T$ be an onto homomorphism of semigroups S and T. Then a map

$$\overline{\theta} : \text{IG}(U) \longrightarrow \text{IG}(V)$$

defined by $\overline{e}\,\overline{\theta} = \overline{e\theta}$ for all $e \in U$, is a well defined homomorphism, where U and V are the biordered sets of S and T, respectively.

IG(E) over simple bands

Let $\alpha = \overline{w_1} \cdots \ \overline{w_k}$ with $w_i \in E_{\gamma_i}$, for $1 \leq i \leq k$. Then

$$(\overline{w_1} \cdots \ \overline{w_k})\ \overline{\theta} = \overline{\gamma_1} \ \cdots \ \overline{\gamma_k}.$$

IG(E) over simple bands

Let $\alpha = \overline{w_1} \cdots \overline{w_k}$ with $w_i \in E_{\gamma_i}$, for $1 \leq i \leq k$. Then

$$(\overline{w_1} \cdots \overline{w_k})\, \overline{\theta} = \overline{\gamma_1} \ \cdots \ \overline{\gamma_k}.$$

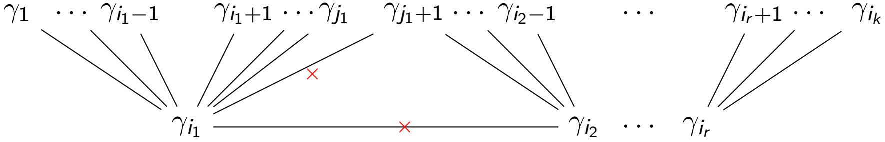
We choose $i_1, j_1, i_2, j_2, \cdots, i_r \in \{1, \cdots, k\}$ in the following way:

IG(E) over simple bands

Let $\alpha = \overline{w_1} \cdots \overline{w_k}$ with $w_i \in E_{\gamma_i}$, for $1 \leq i \leq k$. Then

$$(\overline{w_1} \cdots \overline{w_k})\, \overline{\theta} = \overline{\gamma_1} \cdots \overline{\gamma_k}.$$

We choose $i_1, j_1, i_2, j_2, \cdots, i_r \in \{1, \cdots, k\}$ in the following way:

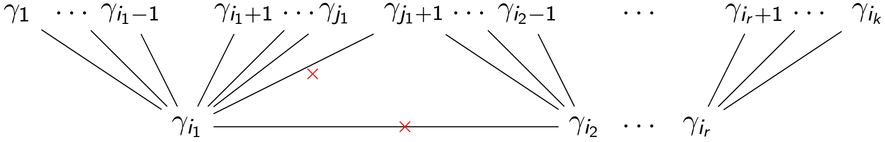


IG(E) over simple bands

Let $\alpha = \overline{w_1} \cdots \overline{w_k}$ with $w_i \in E_{\gamma_i}$, for $1 \leq i \leq k$. Then

$$(\overline{w_1} \cdots \overline{w_k})\,\overline{\theta} = \overline{\gamma_1} \cdots \overline{\gamma_k}.$$

We choose $i_1, j_1, i_2, j_2, \cdots, i_r \in \{1, \cdots, k\}$ in the following way:



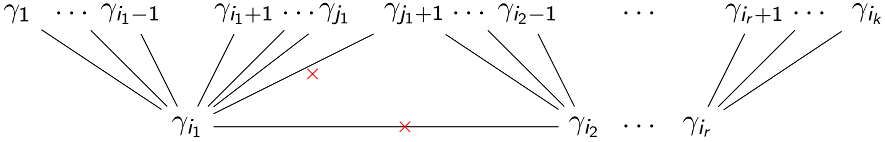
We call $i_1, \cdots, i_r$ are the **significant indices** of α.

IG(E) over simple bands

Let $\alpha = \overline{w_1} \cdots \overline{w_k}$ with $w_i \in E_{\gamma_i}$, for $1 \leq i \leq k$. Then

$$(\overline{w_1} \cdots \overline{w_k})\, \overline{\theta} = \overline{\gamma_1} \cdots \overline{\gamma_k}.$$

We choose $i_1, j_1, i_2, j_2, \cdots, i_r \in \{1, \cdots, k\}$ in the following way:



We call $i_1, \cdots, i_r$ are the **significant indices** of α.

Lemma r and $\gamma_{i_1}, \cdots, \gamma_{i_r}$ are fixed for the equivalence class of α.

IG(E) over simple bands

Lemma Suppose $\alpha = \overline{w_1} \cdots \overline{w_k}$ and $\beta = \overline{x_1} \cdots \overline{x_l}$ with $\alpha \sim \beta$ via single reduction. Suppose that the significant indices of α and β are $i_1, \cdots, i_r$ and $z_1, \cdots, z_r$, respectively. Then $w_1 \cdots w_{i_1} \mathcal{R} x_1 \cdots x_{z_1}$.

IG(E) over simple bands

Lemma Suppose $\alpha = \overline{w_1} \cdots \overline{w_k}$ and $\beta = \overline{x_1} \cdots \overline{x_l}$ with $\alpha \sim \beta$ via single reduction. Suppose that the significant indices of α and β are $i_1, \cdots, i_r$ and $z_1, \cdots, z_r$, respectively. Then $w_1 \cdots w_{i_1} \mathcal{R} x_1 \cdots x_{z_1}$.

Lemma Let E be a simple band and $\overline{x_1} \cdots \overline{x_n} \in \mathrm{IG}(E)$ with normal forms

$$\overline{u_1} \cdots \overline{u_m} = \overline{v_1} \cdots \overline{v_s}.$$

Then $s = m$ and $u_i \mathcal{D} v_i$, for all $i \in [1, m]$.

IG(E) over simple bands

Lemma Suppose $\alpha = \overline{w_1} \cdots \overline{w_k}$ and $\beta = \overline{x_1} \cdots \overline{x_l}$ with $\alpha \sim \beta$ via single reduction. Suppose that the significant indices of α and β are $i_1, \cdots, i_r$ and $z_1, \cdots, z_r$, respectively. Then $w_1 \cdots w_{i_1} \mathcal{R} x_1 \cdots x_{z_1}$.

Lemma Let E be a simple band and $\overline{x_1} \cdots \overline{x_n} \in \mathrm{IG}(E)$ with normal forms

$$\overline{u_1} \cdots \overline{u_m} = \overline{v_1} \cdots \overline{v_s}.$$

Then $s = m$ and $u_i \mathcal{D} v_i$, for all $i \in [1, m]$. In particular, we have

$$u_1 \mathcal{R} v_1 \text{ and } u_m \mathcal{L} v_m.$$

IG(E) over simple bands

Lemma Suppose $\alpha = \overline{w_1} \cdots \overline{w_k}$ and $\beta = \overline{x_1} \cdots \overline{x_l}$ with $\alpha \sim \beta$ via single reduction. Suppose that the significant indices of α and β are $i_1, \cdots, i_r$ and $z_1, \cdots, z_r$, respectively. Then $w_1 \cdots w_{i_1} \mathcal{R} x_1 \cdots x_{z_1}$.

Lemma Let E be a simple band and $\overline{x_1} \cdots \overline{x_n} \in \text{IG}(E)$ with normal forms

$$\overline{u_1} \cdots \overline{u_m} = \overline{v_1} \cdots \overline{v_s}.$$

Then $s = m$ and $u_i \mathcal{D} v_i$, for all $i \in [1, m]$. In particular, we have

$$u_1 \mathcal{R} v_1 \text{ and } u_m \mathcal{L} v_m.$$

Theorem IG(E) over a simple band is a weakly abundant semigroup with the congruence condition.

IG(E) over simple bands

Lemma Suppose $\alpha = \overline{w_1} \cdots \overline{w_k}$ and $\beta = \overline{x_1} \cdots \overline{x_l}$ with $\alpha \sim \beta$ via single reduction. Suppose that the significant indices of α and β are $i_1, \cdots, i_r$ and $z_1, \cdots, z_r$, respectively. Then $w_1 \cdots w_{i_1} \mathcal{R} x_1 \cdots x_{z_1}$.

Lemma Let E be a simple band and $\overline{x_1} \cdots \overline{x_n} \in \mathrm{IG}(E)$ with normal forms

$$\overline{u_1} \cdots \overline{u_m} = \overline{v_1} \cdots \overline{v_s}.$$

Then $s = m$ and $u_i \mathcal{D} v_i$, for all $i \in [1, m]$. In particular, we have

$$u_1 \mathcal{R} v_1 \text{ and } u_m \mathcal{L} v_m.$$

Theorem IG(E) over a simple band is a weakly abundant semigroup with the congruence condition.

IG(E) over strong simple bands

Definition A band E is a **strong simple band** if it is a strong semilattice of right/zero bands.

Definition A band E is a **strong simple band** if it is a strong semilattice of right/zero bands.

Lemma Let $\alpha = \overline{w_1} \cdots \overline{w_k}$ and $\beta = \overline{x_1} \cdots \overline{x_l}$ such that $\alpha \sim \beta$ via single reduction. Suppose that the significant indices of α and β are $i_1, \cdots, i_r$ and $z_1, \cdots, z_r$, respectively. Then

$$\overline{x_1} \cdots \overline{x_{z_l}} = \overline{w_1} \cdots \overline{w_{i_l} u}$$

where $l \in [1, r]$ and $u \in E$.

IG(E) over strong simple bands

Definition A band E is a **strong simple band** if it is a strong semilattice of right/zero bands.

Lemma Let $\alpha = \overline{w_1} \cdots \overline{w_k}$ and $\beta = \overline{x_1} \cdots \overline{x_l}$ such that $\alpha \sim \beta$ via single reduction. Suppose that the significant indices of α and β are $i_1, \cdots, i_r$ and $z_1, \cdots, z_r$, respectively. Then

$$\overline{x_1} \cdots \overline{x_{z_l}} = \overline{w_1} \cdots \overline{w_{i_l} u}$$

where $l \in [1, r]$ and $u \in E$.

Lemma Let $\overline{u_1} \cdots \overline{u_m} = \overline{v_1} \cdots \overline{v_m} \in \mathrm{IG}(E)$ be in normal form. Then
(i) $u_i \,\mathcal{L}\, v_i$ implies $\overline{u_1} \cdots \overline{u_i} = \overline{v_1} \cdots \overline{v_i}$;
(ii) $u_i \,\mathcal{R}\, v_i$ implies $\overline{u_i} \cdots \overline{u_m} = \overline{v_i} \cdots \overline{v_m}$.

IG(E) over strong simple bands

Definition A band E is a **strong simple band** if it is a strong semilattice of right/zero bands.

Lemma Let $\alpha = \overline{w_1} \cdots \overline{w_k}$ and $\beta = \overline{x_1} \cdots \overline{x_l}$ such that $\alpha \sim \beta$ via single reduction. Suppose that the significant indices of α and β are $i_1, \cdots, i_r$ and $z_1, \cdots, z_r$, respectively. Then

$$\overline{x_1} \cdots \overline{x_{z_l}} = \overline{w_1} \cdots \overline{w_{i_l} u}$$

where $l \in [1, r]$ and $u \in E$.

Lemma Let $\overline{u_1} \cdots \overline{u_m} = \overline{v_1} \cdots \overline{v_m} \in \mathrm{IG}(E)$ be in normal form. Then
(i) $u_i \ \mathcal{L} \ v_i$ implies $\overline{u_1} \cdots \overline{u_i} = \overline{v_1} \cdots \overline{v_i}$;
(ii) $u_i \ \mathcal{R} \ v_i$ implies $\overline{u_i} \cdots \overline{u_m} = \overline{v_i} \cdots \overline{v_m}$.

Theorem IG(E) over a strong simple band is abundant.

A key message

Thank you!