Combinatorial semigroups and induced/deduced operators

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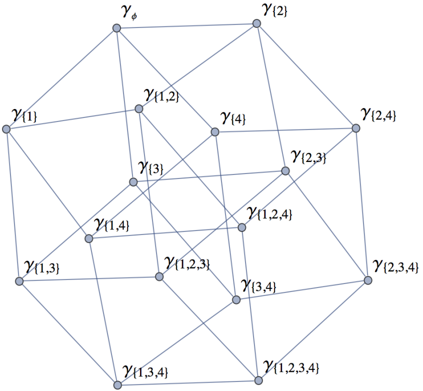
Hypercube $\mathcal{Q}_4$



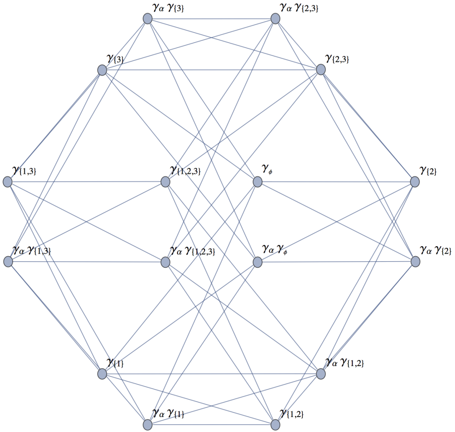
Multi-index notation

- Let $[n] = \{1, 2, \ldots, n\}$ and denote arbitrary, canonically ordered subsets of $[n]$ by capital Roman characters.
- $2^{[n]}$ denotes the *power set* of $[n]$.
- Elements indexed by subsets:

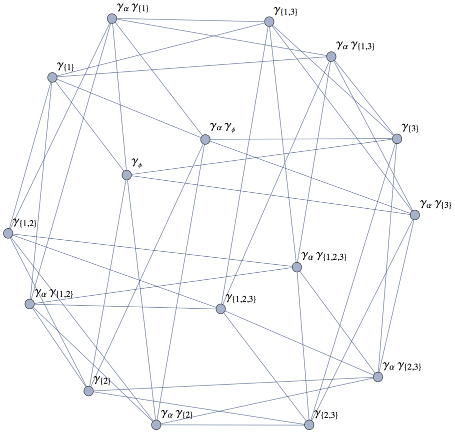
$$\gamma_J = \prod_{j \in J} \gamma_j.$$

- Natural binary representation

Modified hypercubes



Modified hypercubes



Special elements

- $\gamma_\emptyset$ (identity)
- γ_α (commutes with generators, ${\gamma_\alpha}^2 = \gamma_\emptyset$)
- 0_γ ("absorbing element" or "zero")
- "Special" elements do **not** contribute to Hamming weight.

Groups

- Nonabelian – $\mathcal{B}_{p,q}$ "Blade group" (Clifford Lipschitz groups)
  - $\gamma_i\gamma_j = \gamma_\alpha\gamma_j\gamma_i$ $(1 \leq i \neq j \leq p+q)$
  - $$\gamma_i{}^2 = \begin{cases} \gamma_\emptyset & 1 \leq i \leq p, \\ \gamma_\alpha & p+1 \leq i \leq p+q \end{cases}$$
- Abelian – $\mathcal{B}_{p,q}{}^{\text{sum}}$ "Abelian blade group"
  - $\gamma_i\gamma_j = \gamma_j\gamma_i$ $(1 \leq i \neq j \leq p+q)$
  - $$\gamma_i{}^2 = \begin{cases} \gamma_\emptyset & 1 \leq i \leq p, \\ \gamma_\alpha & p+1 \leq i \leq p+q \end{cases}$$

Semigroups

- Nonabelian – "Null blade semigroup" $\mathfrak{Z}_n$
  - $\gamma_i\gamma_j = \gamma_\alpha\gamma_j\gamma_i$ $(1 \leq i \neq j \leq n)$
  - $$\gamma_i{}^2 = \begin{cases} 0 & 1 \leq i \leq n, \\ \gamma_\emptyset & i = \alpha \end{cases}$$
- Abelian – "Zeon semigroup" $\mathfrak{Z}_n{}^{\mathrm{sym}}$
  - $\gamma_i\gamma_j = \gamma_j\gamma_i$ $(1 \leq i \neq j \leq n)$
  - $$\gamma_i{}^2 = \begin{cases} 0 & 1 \leq i \leq n, \\ \gamma_\emptyset & i = \emptyset \end{cases}$$

Passing to semigroup algebra:

- Canonical expansion of arbitrary $u \in \mathcal{A}$:

$$
\begin{aligned}
u &= \sum_{J \in 2^{[n] \cup \{\alpha\}}} u_J \, \gamma_J \\
&= \sum_{J \in 2^{[n]}} {u_J}^{+} \gamma_J + \gamma_\alpha \sum_{J \in 2^{[n]}} {u_J}^{-} \gamma_J .
\end{aligned}
$$

- Naturally graded by Hamming weight (cardinality of J).

Group or Semigroup	Quotient Algebra	Isomorphic Algebra
$\mathcal{B}_{p,q}$	$\mathbb{R}\mathcal{B}_{p,q}/\langle\gamma_\alpha+\gamma_\emptyset\rangle$	$\mathcal{C}\ell_{p,q}$
${\mathcal{B}_{p,q}}^{\text{sym}}$	$\mathbb{R}{\mathcal{B}_{p,q}}^{\text{sym}}/\langle\gamma_\alpha+\gamma_\emptyset\rangle$	${\mathcal{C}\ell_{p,q}}^{\text{sym}}$
$\mathfrak{Z}_n$	$\mathbb{R}\mathfrak{Z}_n/\langle 0_\gamma, \gamma_\alpha+\gamma_\emptyset\rangle$	$\bigwedge\mathbb{R}^n$
${\mathfrak{Z}_n}^{\text{sym}}$	$\mathbb{R}{\mathfrak{Z}_n}^{\text{sym}}/\langle 0_\gamma\rangle$	${\mathcal{C}\ell_n}^{\text{nil}}$

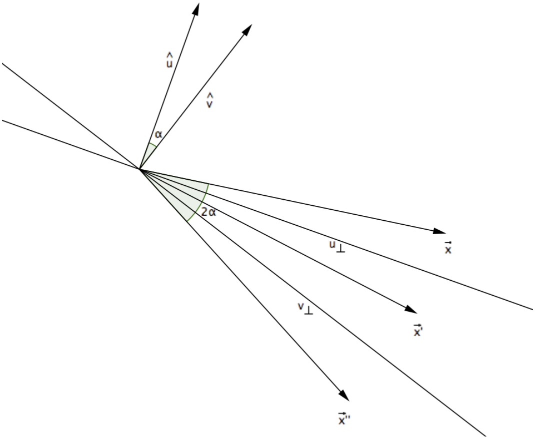
Idea: Induced Operators

1. Let V be the vector space spanned by generators $\{\gamma_j\}$ of (semi)group $\mathcal{S}$.
2. Let A be a linear operator on V.
3. A naturally induces an operator $\mathfrak{A}$ on the semigroup algebra $\mathbb{R}\mathcal{S}$ according to action (multiplication, conjugation, etc.) on $\mathcal{S}$.
   - $\mathfrak{A}(\gamma_J) := \prod_{j\in J} A(\gamma_j)$

The Clifford algebra $\mathcal{C}\ell_{p,q}$

1. Real, associative algebra of dimension 2^n.
2. Generators $\{\mathbf{e}_i : 1 \leq i \leq n\}$ along with the unit scalar $\mathbf{e}_\emptyset = 1 \in \mathbb{R}$.
3. Generators satisfy:
   - $[\mathbf{e}_i, \mathbf{e}_j] := \mathbf{e}_i\,\mathbf{e}_j + \mathbf{e}_j\,\mathbf{e}_i = 0$ for $1 \leq i \neq j \leq n$,
   - $$\mathbf{e}_i{}^2 = \begin{cases} 1 & \text{if } 1 \leq i \leq p, \\ -1 & \text{if} p+1 \leq i \leq p+q. \end{cases}$$

Rotations & Reflections: $x \mapsto uvxvu$



Hyperplane Reflections

1. Product of orthogonal vectors is a *blade*.
2. Given unit blade $\mathfrak{u} \in \mathcal{C}\ell_Q(V)$, where Q is positive definite.
3. The map $\mathbf{x} \mapsto \mathfrak{u}\mathbf{x}\mathfrak{u}^{-1}$ represents a composition of hyperplane reflections across pairwise-orthogonal hyperplanes.
4. This is group action by conjugation.
5. Each vertex of the hypercube underlying the Cayley graph corresponds to a hyperplane arrangement.

Blade conjugation

1. $\mathfrak{u} \in \mathcal{B}\ell_{p,q} \simeq \mathcal{C}\ell_Q(V)$ a blade.
2. $\Phi_\mathfrak{u}(\mathbf{x}) := \mathfrak{u}\mathbf{x}\mathfrak{u}^{-1}$ is a Q-orthogonal transformation on V.
3. $\Phi_\mathfrak{u}$ induces $\varphi_\mathfrak{u}$ on $\mathcal{C}\ell_Q(V)$.
4. The operators are self-adjoint w.r.t. $\langle \cdot, \cdot \rangle_Q$; i.e., they are *quantum random variables*.
5. Characteristic polynomial of $\Phi_\mathfrak{u}$ generates *Kravchuk polynomials*.

Blade conjugation

1. Conjugation operators allow factoring of blades.
   - Eigenvalues ± 1
   - Basis for each eigenspace provides factorization of corresponding blade.
2. Quantum random variables obtained at every level of induced operators.
   - $\varphi^{(\ell)}$ is self-adjoint w.r.t. Q-inner product for each $\ell = 1, \ldots, n$.
3. Kravchuk polynomials appear in traces at every level.
4. Kravchuk matrices represent blade conjugation operators (in most cases[1]).

[1] G.S. Staples, *Kravchuk Polynomials & Induced/Reduced Operators on Clifford Algebras*, Preprint (2013).

More generally...

1. Suppose X is a linear operator on V.
2. Suppose $I, J \in 2^{|V|}$ with $|I| = |J| = \ell$.
3. Then, $\langle v_I | \mathfrak{X}^{(\ell)} | v_J \rangle = \det(X_{IJ})$.
   - Here, X_{IJ} is the submatrix of X formed from the rows indexed by I and the columns indexed by J.
   - This holds for $\mathcal{C}\ell_Q(V)$ as well as $\bigwedge V$. In the latter case, $\mathfrak{X}$ is block diagonal.

The zeon algebra $\mathcal{C}\ell_n{}^{\mathrm{nil}}$

1. Real, associative algebra of dimension 2^n.
2. Generators $\{\zeta_i : 1 \leq i \leq n\}$ along with the unit scalar $\zeta_\emptyset = 1 \in \mathbb{R}$.
3. Generators satisfy:
   - $[\zeta_i, \zeta_j] := \zeta_i \zeta_j - \zeta_j \zeta_i = 0$ for $1 \leq i, j \leq n$,
   - $\zeta_i \zeta_j = 0 \Leftrightarrow i = j$.

Zeons

1. Applications in combinatorics, graph theory, quantum probability explored in monograph by Schott & Staples [2]. Based on papers by Staples and joint work with Schott.
2. Induced maps appear in work by Feinsilver & McSorley [3]

[2] *Operator Calculus on Graphs (Theory and Applications in Computer Science)*, Imperial College Press, London, 2012

[3] P. Feinsilver, J. McSorley, Zeons, permanents, the Johnson scheme, and generalized derangements, *International Journal of Combinatorics*, vol. 2011, Article ID 539030, 29 pages, 2011. doi:10.1155/2011/539030

Adjacency matrices

1. Let $G = (V, E)$ be a graph on n vertices.
2. Let A denote the adjacency matrix of G, viewed as a linear transformation on the vector space generated by $V = \{v_1, \ldots, v_n\}$.
3. $\mathfrak{A}^{(k)}$ denotes the multiplication-induced operator on the grade-k subspace of the semigroup algebra $\mathcal{C}\ell_V{}^{\mathrm{nil}}$.

Theorem

For fixed subset $I \subseteq V$, let X_I denote the number of disjoint cycle covers of the subgraph induced by I. Similarly, let M_J denote the number of perfect matchings on the subgraph induced by $J \subseteq V$ (nonzero only for J of even cardinality). Then,

$$\mathrm{tr}(\mathfrak{A}^{(k)}) = \sum_{\substack{I \subset V \\ |I|=k}} \sum_{J \subseteq I} X_{I \setminus J} M_J.$$

Sketch of Proof

1. $\langle v_J|\mathfrak{A}^{(k)}|v_J\rangle = \mathrm{per}(A_J + \mathcal{I})$, where A_J is the adjacency matrix of the subgraph induced by v_J.
2. $$\mathrm{per}(A_J + \mathcal{I}) := \sum_{\sigma \in \mathrm{S}_{|J|}} \prod_{j=1}^{|J|} a_{j\,\sigma(j)}$$
3. Each permutation has a unique factorization into disjoint cycles. Each product of 2-cycles corresponds to a perfect matching on a subgraph. Cycles of higher order in $\mathrm{S}_{|J|}$ correspond to cycles in the graph.

Generating Function

Let A be the adjacency matrix of a graph. Letting $f(t) := \mathrm{per}(A + t\mathcal{I})$, one finds that the coefficient of t^{n-k} satisfies

$$\langle f(t), t^{(n-k)} \rangle = \mathrm{tr}(\mathfrak{A}^{(k)}).$$

Hence,

$$f^{(n-k)}(0) = (n-k)!\mathrm{tr}(\mathfrak{A}^{(k)}).$$

Nilpotent Adjacency Operator

Let $G = (V, E)$ be a graph on n vertices, and let A be the adjacency matrix of G.

1. $\{\zeta_i : 1 \leq i \leq n\}$ generators of $\mathcal{C}\ell_n{}^{\text{nil}}$.
2. The *nilpotent adjacency operator* associated with G is an operator $\mathfrak{A}$ on $(\mathcal{C}\ell_n{}^{\text{nil}})^n$ induced by A via

$$\langle v_i|\mathfrak{A}|v_j\rangle = \begin{cases} \zeta_j \text{ if } (v_i, v_j) \in E(G), \\ 0 \text{ otherwise.} \end{cases}$$

Form of $\mathfrak{A}^k$

Theorem

Let $\mathfrak{A}$ be the nilpotent adjacency operator of an n-vertex graph G. For any $k > 1$ and $1 \leq i, j \leq n$,

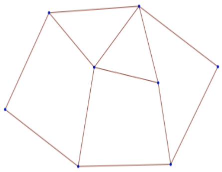
$$\left\langle v_i|\mathfrak{A}^k|v_j\right\rangle = \sum_{\substack{I\subseteq V\\|I|=k}} \omega_I \zeta_I, \tag{1}$$

where ω_I denotes the number of k-step walks from v_i to v_j revisiting initial vertex v_i exactly once when $i \in I$ and visiting each vertex in I exactly once when $i \notin I$.

Idea

1. $\mathfrak{A}$ is represented by a **nilpotent adjacency matrix**.
2. Powers of the nilpotent adjacency matrix “sieve-out” the self-avoiding structures in the graph.
3. “Automatic pruning” of tree structures.

Example



$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \zeta_{\{2\}} & 0 & \zeta_{\{4\}} & 0 & 0 & 0 & 0 \\ \zeta_{\{1\}} & 0 & 0 & 0 & 0 & \zeta_{\{6\}} & \zeta_{\{7\}} & 0 \\ 0 & 0 & 0 & 0 & \zeta_{\{5\}} & \zeta_{\{6\}} & 0 & 0 \\ \zeta_{\{1\}} & 0 & 0 & 0 & \zeta_{\{5\}} & 0 & \zeta_{\{7\}} & \zeta_{\{8\}} \\ 0 & 0 & \zeta_{\{3\}} & \zeta_{\{4\}} & 0 & 0 & 0 & \zeta_{\{8\}} \\ 0 & \zeta_{\{2\}} & \zeta_{\{3\}} & 0 & 0 & 0 & 0 & \zeta_{\{8\}} \\ 0 & \zeta_{\{2\}} & 0 & \zeta_{\{4\}} & 0 & 0 & 0 & \zeta_{\{8\}} \\ 0 & 0 & 0 & \zeta_{\{4\}} & \zeta_{\{5\}} & \zeta_{\{6\}} & \zeta_{\{7\}} & 0 \end{pmatrix}$$

Cycles from $\mathfrak{A}^k$

Corollary

For any $k \geq 3$ and $1 \leq i \leq n$,

$$\left\langle v_i | \mathfrak{A}^k | v_i \right\rangle = \sum_{\substack{I \subseteq V \\ |I| = k}} \xi_I \zeta_I, \tag{2}$$

where ξ_I denotes the number of k-cycles on vertex set I based at $i \in I$.

Flexibility

1. Convenient for symbolic computation
2. Easy to consider other self-avoiding structures (trails, circuits, partitions, etc.)
3. Extends to random graphs, Markov chains, etc.
4. Sequences of operators model graph processes
5. The operators themselves generate finite semigroups.

Idea: Reduced Operators

1. Consider operator $\mathfrak{A}$ on the semigroup algebra $\mathbb{R}\mathcal{S}$.
2. Let V be the vector space spanned by generators $\{\gamma_j\}$ of (semi)group $\mathcal{S}$.
3. If $\mathfrak{A}$ is induced by an operator A on V, then $A = \mathfrak{A}\Big|_V$ is the operator on V *deduced* from $\mathfrak{A}$.
4. Let $V_* = \mathbb{R} \oplus V$ be the *paravector space* associated with V.
5. $\mathfrak{A}$ naturally *reduces by grade* to an operator A' on V_*.

Grade-reduced operators

1. Paravector space $V_* = \mathbb{R} \oplus V$ spanned by ordered basis $\{\varepsilon_0, \ldots, \varepsilon_n\}$
2. Operator A on $V_* = \mathbb{R} \oplus V$ is *grade-reduced* from $\mathfrak{A}$ if its action on the basis of V_* satisfies

$$\langle \varepsilon_i | A | \varepsilon_j \rangle = \sum_{\substack{\sharp\mathfrak{a}=i \\ \sharp\mathfrak{b}=j}} \langle \mathfrak{a} | \mathfrak{A} | \mathfrak{b} \rangle,$$

where the sum is taken over blades in some fixed basis of $\mathbb{R}\mathcal{S}$. Write $\mathfrak{A} \searrow A$.

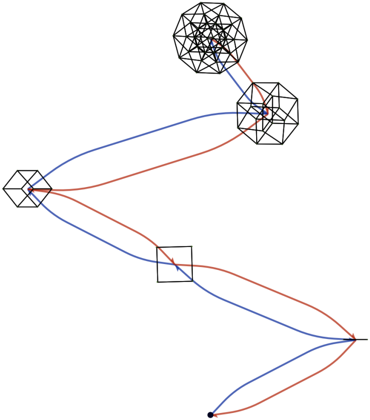
Properties & Interpretation

1. In $\mathcal{C}\ell_Q(V)$, Kravchuk matrices and symmetric Kravchuk matrices arise.
2. Over zeons, graph-theoretic interpretations arise. Suppose A is the adjacency matrix of graph G and that $A \nearrow \mathfrak{A} \searrow A'$. Then,
   - $\langle \varepsilon_k | A' | \varepsilon_k \rangle = \operatorname{tr}(\mathfrak{A}^{(k)}) = \sum_{\substack{I \subset V \\ |I|=k}} \sum_{J \subseteq I} X_{I \setminus J} M_J.$
   - $\operatorname{tr}(A') = \sum_{k=0}^{n} \sum_{\substack{I \subset V \\ |I|=k}} \sum_{J \subseteq I} X_{I \setminus J} M_J$

Operator Calculus (OC)

1. Lowering operator Λ
   - differentiation
   - annihilation
   - deletion
2. Raising operator Ξ
   - integration
   - creation
   - addition/insertion

Raising & Lowering



OC & Clifford multiplication

1. Left lowering $\Lambda_{\mathbf{x}}$: $u \mapsto \mathbf{x} \lrcorner u$
2. Right lowering $\hat{\Lambda}_{\mathbf{x}}$: $u \mapsto u \llcorner \mathbf{x}$
3. Left raising $\Xi_{\mathbf{x}}$: $u \mapsto \mathbf{x} \wedge u$
4. Right raising $\hat{\Xi}_{\mathbf{x}}$: $u \mapsto u \wedge \mathbf{x}$
5. Clifford product satisfies

$$\mathbf{x}u = \Lambda_{\mathbf{x}}\mathbf{u} + \hat{\Xi}_{\mathbf{x}}u$$
$$u\mathbf{x} = \hat{\Lambda}_{\mathbf{x}}\mathbf{u} + \Xi_{\mathbf{x}}u$$

OC & blade conjugation

1. Given a blade $\mathfrak{u} \in \mathcal{C}\ell_Q(V)$;
2. Extend lowering, raising by associativity to blades, i.e., $\Lambda_{\mathfrak{u}}$, $\Xi_{\mathfrak{u}}$, etc.
3. Operator calculus (OC) representation of conjugation operator $\varphi_{\mathfrak{u}}$, $x \mapsto \mathfrak{u} x \mathfrak{u}^{-1}$, is

$$\varphi_{\mathfrak{u}} \simeq \Lambda_{\mathfrak{u}} \Xi_{\mathfrak{u}^{-1}} + \hat{\hat{\Xi}}_{\mathfrak{u}} \hat{\Lambda}_{\mathfrak{u}^{-1}}.$$

Motivation

Graphs → Algebras

Processes on Algebras → Processes on Graphs

Random walks & stochastic processes

1. Walks on hypercubes $\leftrightarrow$ addition-deletion processes on graphs
2. Walks on “signed hypercubes” $\leftrightarrow$ multiplicative processes on Clifford algebras

Random walks & stochastic processes

1. Partition-dependent stochastic integrals. (Staples, Schott & Staples)
2. Random walks on hypercubes can be modeled by raising and lowering operators. (Staples)
3. Random walks in Clifford algebras (Schott & Staples)

Graph processes as algebraic processes

The idea is to encode the entire process using (nilpotent) adjacency operators and use projections to recover information about graphs at different steps of the process:

- Expected numbers of **cycles**
- Probability of **connectedness**
- Expected numbers of **spanning trees**
- Determine size of maximally connected **components**
- Expected **time** at which graph becomes **connected/disconnected**
- **Limit** theorems

Addition-deletion processes via hypercubes

1. Graph $\mathcal{G}_{[n]}$ on vertex set $V = [n]$ with predetermined topology.
2. Markov chain (X_k) on power set of V.
   - Family of functions $f_\ell : 2^{[n]} \to [0,1]$ such that for each $I \in 2^{[n]}$,
   $$\sum_{\ell=1}^{n} f_\ell(I) = 1.$$
   - $$\mathbb{P}(X_k = I | X_{k-1} = J) = \begin{cases} f_\ell(J) & I \triangle J = \{\ell\}, \\ 0 & \text{otherwise.} \end{cases}$$

Walks on $\mathcal{Q}_n$

1. Walk on $\mathcal{Q}_n$ corresponds to graph process $(\mathcal{G}(U_n) : n \in \mathbb{N}_0)$.
2. Each vertex of $\mathcal{Q}_n$ is uniquely identified with a graph.
3. Adding a vertex corresponds to combinatorial raising.
4. Deleting a vertex corresponds to combinatorial lowering.

Addition-deletion processes via hypercubes

1. Corresponds to Markov chain (ξ_t) on a commutative algebra by $U \mapsto \varsigma_U$, with multiplication $\varsigma_U \varsigma_V = \varsigma_{U \triangle V}$.
2. Markov chain induced on the state space of all vertex-induced subgraphs. I.e.,

$$\mathcal{S} = \{\mathcal{G}_U : U \subseteq V\}$$

3. Let Ψ_U denote the *nilpotent adjacency operator* of the graph $\mathcal{G}_U$.

Addition-deletion processes via hypercubes

1. Well-defined mapping $\varsigma_U \mapsto \varsigma_U \Psi_U$
2. Expected value at time ℓ:

$$\langle \xi_\ell \rangle = \sum_{U \in 2^{[n]}} \mathbb{P}(\xi_\ell = \varsigma_U)\varsigma_U.$$

3. Define notation:

$$\Psi_{\langle \xi_\ell \rangle} \sum_{U \in 2^{[n]}} \mathbb{P}(\xi_\ell = \varsigma_U)\varsigma_U \Psi_U.$$

Paths Lemma

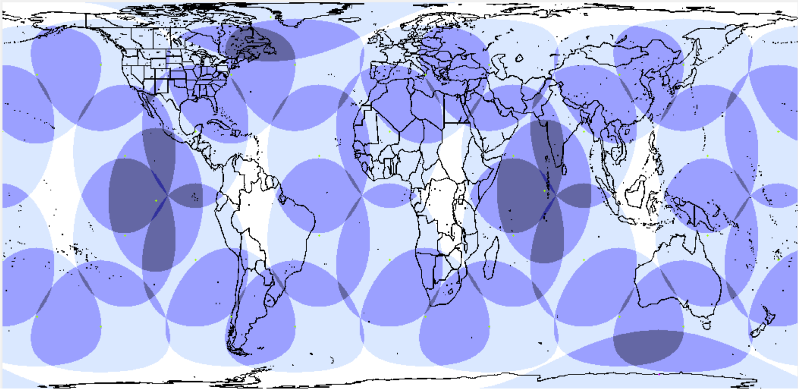
Given vertices $v_i, v_j \in V$, the expected number of k-paths v_i to v_j at time ℓ in the addition-deletion process $(\mathcal{G}_t)$ is given by

$$\mathbb{E}(|\{k\text{-paths } v_i \to v_j \text{ at time } \ell\}|) = \zeta_{\{i\}} \sum_{U \in 2^{[n]}} \langle v_i | \langle \Psi_{\langle \xi_\ell \rangle}, \varsigma_U \rangle^k | v_j \rangle.$$

Networks

- Wireless sensor networks (Ben Slimane, Nefzi, Schott, & Song)
- Satellite communications (w. Cruz-Sánchez, Schott, & Song)
- Mobile ad-hoc networks

Satellite communications



Shortest Paths: Motivation

- Goal: To send a data packet from node v_{initial} to node v_{term} quickly and reliably.
- In order to route the packet efficiently, you need to know something about the paths from v_{initial} to v_{term} in the graph.
- When the graph is changing, a sequence of nilpotent adjacency operators can be used. [4]

[4] H. Cruz-Sánchez, G.S. Staples, R. Schott, Y-Q. Song, Operator calculus approach to minimal paths: Precomputed routing in a store-and-forward satellite constellation, *Proceedings of IEEE Globecom 2012, Anaheim, USA, December 3-7*, 3438–3443.

To be continued...

THANKS FOR YOUR ATTENTION!

More on Clifford algebras, operator calculus, and stochastic processes

- http://www.siue.edu/~sstaple

More on Clifford algebras, graph theory, and stochastic processes

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