Random bipartite graphs

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Random graphs

Definition

A κ-random graph is a graph (V, E) such that $|V| = \kappa$ that satisfies the following extension property:

$$\forall U, W \in [V]^{<\kappa}(U \cap W = \emptyset \Rightarrow \exists v \in V(\forall u \in U\ vu \in E \wedge \forall w \in W\ vw \notin E)).$$

Rado graph - the unique $\aleph_0$-random graph.

Related structures: random digraphs, random tournaments, etc.

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Random bigraphs

Definition

(κ, λ)-bigraph is a structure $G = (X, Y, E)$, where $(X \cup Y, E)$ is a digraph such that $|X| = \kappa$, $|Y| = \lambda$ and $E \subseteq \{xy : x \in X, y \in Y\}$.

We call X the left side, and Y the right side.

$\Gamma^G_{U,W} = \{x \in X : \forall u \in U \; xu \in E \wedge \forall w \in W \; xw \notin E\}$

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$$\forall U, W \in [Y]^{<\mu}\ (U \cap W = \emptyset \Rightarrow \Gamma^G_{U,W} \neq \emptyset).$$

Definition

If $\mu \leq \kappa$, a (κ, λ)-bigraph (X, Y, E) is (κ, λ, μ)-dense if

$$\forall U, W \in [X]^{<\mu}\ (U \cap W = \emptyset \Rightarrow \exists y \in Y (\forall u \in U\ uy \in E\ \wedge\ \forall w \in W\ wy \notin E)).$$

If G satisfies both conditions we will call it (κ, λ, μ)-random dense.
A $(\kappa, \lambda, \aleph_0)$-random bigraph is called just (κ, λ)-random.

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Lemma

(a) In a (κ, λ, μ)-random bigraph (X, Y, E) we can find for every disjoint $U, W \in [Y]^{<\mu}$ μ-many vertices $x \in X$ that satisfy $xu \in E$ for all $u \in U$ and $xw \notin E$ for all $w \in W$.
(b) In a (κ, λ, μ)-dense bigraph (X, Y, E) we can find for every disjoint $U, W \in [X]^{<\mu}$ μ-many vertices $y \in Y$ that satisfy $uy \in E$ for all $u \in U$ and $wy \notin E$ for all $w \in W$.

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Independent and dense families

κ-random graph: $\forall U, W \in [V]^{<\kappa}(U \cap W = \emptyset \Rightarrow \exists v \in V(\forall u \in U\ vu \in E \wedge \forall w \in W\ vw \notin E))$.

Definition

Let $\mu \leq \lambda$. A family $\mathcal{A} = \{A_\alpha : \alpha < \lambda\}$ of subsets of κ is called (κ, λ, μ)-independent if

$$\forall U, W \in [\lambda]^{<\mu}(U \cap W = \emptyset \Rightarrow \bigcap_{\alpha \in U} A_\alpha \cap \bigcap_{\alpha \in W} (\kappa \setminus A_\alpha) \neq \emptyset).$$

The connection

Let $\mathcal{A} = \{A_\alpha : \alpha < \lambda\}$ be a (κ, λ, μ)-independent family. Let X and Y be disjoint sets of cardinalities κ and λ respectively. We enumerate them: $X = \{x_\beta : \beta < \kappa\}$, $Y = \{y_\alpha : \alpha < \lambda\}$, and define the relation $E \subseteq X \times Y$: let $x_\beta y_\alpha \in E$ iff $\beta \in A_\alpha$. Then (X, Y, E) is a (κ, λ, μ)-random bigraph.

On the other hand, let $G = (X, Y, E)$ be a (κ, λ, μ)-random bigraph. We enumerate $X = \{x_\beta : \beta < \kappa\}$ and $Y = \{y_\alpha : \alpha < \lambda\}$ and define, for each $\alpha \in \lambda$, $A_\alpha = \{\beta \in \kappa : x_\beta y_\alpha \in E\}$. Then $\{A_\alpha : \alpha < \lambda\}$ is a (κ, λ, μ)-independent family.

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Robustness

Lemma

Every bigraph obtained from a (κ, λ, μ)-random bigraph (X, Y, E) by

(a) adding $\leq \kappa$ vertices to X (connected to arbitrary vertices from Y)

(b) removing $< \mu$ vertices from X

(c) removing $< \lambda$ vertices from Y

(d) replacing $< \mu$ edges with non-edges and $< \mu$ non-edges with edges

is also a (κ, λ, μ)-random bigraph.

Lemma

Let μ be a regular cardinal. Every bigraph obtained from a (κ, λ, μ)-random dense bigraph by deleting $< \mu$ edges from each vertex is also a (κ, λ, μ)-random dense bigraph.

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Existence, uniqueness, homogeneity

Fact

If $\kappa^{<\mu} = \kappa$ then there is a $(\kappa, 2^\kappa, \mu)$-random bigraph.

Existence, uniqueness, homogeneity

Theorem (Goldstern, Grossberg, Kojman, 1996)

(a) There is exactly one (up to isomorphism) $(\aleph_0, \aleph_0)$-random dense bigraph, and it is homogeneous.

(b) Every homogeneous (κ, λ)-bigraph which is neither empty nor complete is either a perfect matching or its complement or a (κ, λ)-random dense bigraph (of course, when $\kappa \neq \lambda$, only the latter option remains).

(c) There is a $(\kappa, 2^\kappa)$-random dense bigraph for every infinite cardinal κ.

(d) $(\neg\mathrm{CH} \wedge \mathrm{MA})$ For every $\kappa < \mathfrak{c}$ there is unique $(\aleph_0, \kappa)$-random dense bigraph up to isomorphism.

(e) $(2^{\kappa^+} > 2^\kappa)$ There are 2^{κ^+}-many nonisomorphic (κ, κ^+)-random dense bigraphs.

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Universality

Theorem

Every (κ_1, λ_1)-bigraph for $\kappa_1 \leq \mu$ and $\lambda_1 < \mu$ can be embedded in any (κ, λ, μ)-random bigraph.

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(b) Every (κ, κ, κ)-random dense bigraph has a 1-factorization, i.e. its set of edges can be partitioned into disjoint perfect matchings.

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A partition property for graphs

$\mathcal{P}$: for every partition of the set of vertices of G into finitely many pieces at least one of the induced graphs is isomorphic to G.

Theorem (Cameron)

The only countable graphs with the property $\mathcal{P}$ up to isomorphism are the empty graph, the complete graph and the Rado graph.

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Theorem (Bonato, Cameron, Delić, 2000)

The only countable tournaments with the property $\mathcal{P}$ up to isomorphism are the random tournament, and tournaments ω^α and $(\omega^\alpha)^*$ for $0 < \alpha < \omega_1$.

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$\mathcal{P}$: for every partition of the set of vertices of G into finitely many pieces at least one of the induced sub-bigraphs is isomorphic to G.

A partition property for bigraphs

$\mathcal{P}'$: for every partition of the set of vertices of G into finitely many pieces that each induce $(\aleph_0, \aleph_0)$-bigraphs at least one of the induced sub-bigraphs is isomorphic to G.

Lemma

Let μ be a regular cardinal and $\nu < \mu$. Let $\{V_\gamma : \gamma < \nu\}$ be a partition of the set of vertices of (κ, λ, μ)-random bigraph such that each V_γ has at least μ vertices on each side. Then at least one of the induced sub-bigraphs is $(\kappa_1, \lambda_1, \mu)$-random for some $\kappa_1 \leq \kappa$ and $\lambda_1 \leq \lambda$.

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The $(\aleph_0, \aleph_0)$-random dense bigraph does not satisfy $\mathcal{P}'$.

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Example

$S = (X, Y, E)$ is defined by $X = \{x_n : n \in \omega\}$, $Y = \{y_n : n \in \omega\}$ and $E = \{x_n y_0 : n \in \omega\}$. S, S^* and their complements have $\mathcal{P}'$.

Theorem

The only $(\aleph_0, \aleph_0)$-bigraphs with the property $\mathcal{P}'$ up to isomorphism are the empty $(\aleph_0, \aleph_0)$-bigraph, the complete $(\aleph_0, \aleph_0)$-bigraph, the bigraphs S and S^* and their complements.

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More partition properties

Theorem

Let $X = X_0 \cup X_1$ and $Y = Y_0 \cup Y_1$ be partitions of sides of the $(\aleph_0, \aleph_0)$-random dense bigraph G into infinite subsets.

(a) There is $i \in \{0, 1\}$ such that the sub-bigraph induced by $X_i \cup Y_i$ is $(\aleph_0, \aleph_0)$-random.

(b) There is $j \in \{0, 1\}$ such that the sub-bigraph induced by $X_j \cup Y_j$ is $(\aleph_0, \aleph_0)$-dense.

(c) There are $i, j \in \{0, 1\}$ such that the sub-bigraph induced by $X_i \cup Y_j$ is $(\aleph_0, \aleph_0)$-random dense and hence isomorphic to G.

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(b) There is $j \in \{0, 1\}$ such that the sub-bigraph induced by $X_j \cup Y_j$ is $(\aleph_0, \aleph_0)$-dense.
(c) There are $i, j \in \{0, 1\}$ such that the sub-bigraph induced by $X_i \cup Y_j$ is $(\aleph_0, \aleph_0)$-random dense and hence isomorphic to G.

More partition properties

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