Sierpiński rank of groups and semigroups

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Sierpiński Rank

The **Sierpiński rank** of a semigroup S is the least $N \in \mathbb{N}$ such that every countable set of elements in S is contained in an N-generated subsemigroup, if such an N exists.

Theorem (Sierpiński ’35)

The Sierpiński rank of Ω^Ω *is* 2.

The Hyde-Star-Trek-Analogy proof...

Theorem (Sierpiński '35)

The Sierpiński rank of Ω^Ω *is* 2.

Proof. Suppose Ω is countable and identify Ω with the eventually constant sequences over $\mathbb{N}$.

- let $f_1, f_2, \ldots \in \Omega^\Omega$
- define:

$$\begin{aligned} a : (x_0, x_1, \ldots) &\mapsto (0, x_0, x_1, \ldots) \\ b : (x_0, x_1, \ldots) &\mapsto (x_0 + 1, x_1, \ldots) \\ c : (x_0, x_1, \ldots) &\mapsto (x_1, x_2, \ldots) f_{x_0} \end{aligned}$$

- $(x_0, x_1, \ldots) \overset{a}{\mapsto} (0, x_0, x_1, \ldots) \overset{b^n}{\mapsto} (n, x_0, x_1, \ldots) \overset{c}{\mapsto} (x_0, x_1, \ldots) f_n$

the previous line implies that: $ab^n c = f_n$ for all $n \geq 1$

- if we set $f_0 = b$, then $b = f_0 = ac$ and so $f_n = a(ac)^n c$. □

Further examples...

- an infinite semigroup has Sierpiński rank 1 if and only if it is isomorphic to $(\mathbb{N}, +)$.
- $\exists$ uncountable semigroups with Sierpiński rank n for every $n \geq 2$.
- the **increasing functions** on the unit interval $[0, 1]$ have Sierpiński rank 3.
- the endomorphism monoid of the random graph has Sierpiński rank 2 or 3.2.
- the **partition monoid** has Sierpiński rank ≤ 42.
- the **surjective** functions on Ω where $|\Omega| = \aleph_n$ $(n \in \mathbb{N})$ have Sierpiński rank $n^2/2 + 9n/2 + 7$.
- (Easty) the **factorizable part of the dual symmetric inverse monoid** on Ω where $|\Omega| = \aleph_n$ $(n \in \mathbb{N})$ has Sierpiński rank $n^2/2 + 3n/2 + 4$.
- the **Baer-Levi semigroups** and **order-preserving** functions on $\mathbb{N}$ have infinite Sierpiński rank.

Sierpiński rank for groups

The **Sierpiński rank** of an infinite **semigroup~~semi~~groupgroupinverse semigroup** S is the least $N \in \mathbb{N}$ such that every countable set of elements in S is contained in an N-generated **subsemigroupsub~~semi~~groupsubgroupsubsemigroupinverse subsemigroupsubsemigroup**, if such an N exists.

- (Galvin '95) If $f_0, f_1, \ldots \in \mathrm{Sym}(\Omega)$, then there exist $a, b \in \mathrm{Sym}(\Omega)$ with **finite order** such that $f_0, f_1, \ldots \in \langle\, a, b \,\rangle$.
- (Truss) If G is the group of homeomorphisms of the Cantor space, $\mathbb{Q}$, or $\mathbb{R} \setminus \mathbb{Q}$, then G has Sierpiński rank 2.
- (Calegari-Freedman-Cornulier '06) The group of homeomorphisms of the euclidean m-sphere have finite Sierpiński rank.
- the **symmetric inverse** and the **dual symmetric inverse** monoids have Sierpiński rank 2.

What did we really prove?

Strongly distorted

If $f_0, f_1, \ldots \in \Omega^\Omega$, then there exist $a, b \in \Omega^\Omega$ such that $f_n \in \langle\, a, b \,\rangle$ for all n **and the length of the product is at most $2n+2$**.

Universal sequence

If $f_0, f_1, \ldots \in \Omega^\Omega$, then there exist $a, b \in \Omega^\Omega$ such that $f_n = a(ab)^n b$. (Not only is $f_n \in \langle\, a, b \,\rangle$ but **we specified a product of a and b**.)

These are 3 distinct notions.

For example, the **surjective** functions on Ω where $|\Omega| = \aleph_5$ have Sierpiński rank 42 but they do not satisfy either of the properties above.

Bergman's Property

Definition (Bergman's Property)

A semigroup S has this property if given any generating set U for S there exists $n \in \mathbb{N}$ such that $S = U \cup U^2 \cup \cdots \cup U^n$.

Theorem (Bergman '05)

The symmetric group on any infinite set has my property!

Lots of groups have Bergman's property:

- (Shelah '76) an uncountable group G with $U^{240} = G$ for all generating sets U
- (Droste & Göbel '05) the order-automorphisms of the rationals, the homeomorphisms of the irrationals, ...

Some don't:

- free groups and finitely generated infinite groups
- (Droste & Göbel '05) $\{ f \in \mathrm{Sym}(\mathbb{Q}) : \exists k \ |i - (i)f| \leq k \}$

The connection?

Theorem (Galvin '95)

If $f_0, f_1, \ldots \in \mathrm{Sym}(\Omega)$*, then* $\exists a, b \in \mathrm{Sym}(\Omega)$ *s.t.* $f_n \in \langle\, a, b \,\rangle$ *and* $|f_n| \leq 4n + 18$*.*

Lemma (Droste & Göbel '06)

Let S *be a non-f.g. semigroup where every* $\Psi : S \to \mathbb{N}$ *satisfying*

$$(st)\Psi \leq (s)\Psi + (t)\Psi + k_\Psi \qquad \forall s, t \in S$$

is bounded above. Then S *has Bergman's Property.*

Proof of Bergman's Theorem.

- Suppose $\Psi : \mathrm{Sym}(\Omega) \longrightarrow \mathbb{N}$ is an unbounded function satisfying

$$(st)\Psi \leq (s)\Psi + (t)\Psi + k_\Psi$$

- there exist $f_0, f_1, \ldots \in \mathrm{Sym}(\Omega)$ such that $(f_n)\Psi > n^2$ for all $n \in \mathbb{N}$
- $\exists a, b \in \mathrm{Sym}(\Omega)$ s.t. $f_n \in \langle\, a, b \,\rangle$ and $|f_n| \leq 4n + 18$ and so

$$(f_n)\Psi \leq (4n+18) \cdot \left(\max\{(a)\Psi, (b)\Psi\} + k_\Psi \right) < n^2$$

  for sufficiently large n, a contradiction
- so Ψ is bounded and $\mathrm{Sym}(\Omega)$ has Bergman's Property. □

Corollary

Every strongly distorted infinite semigroup has Bergman's property.

The random graph

Theorem (Erdős and Rényi '63)

There is a countable graph R such that a random countable graph (edges chosen independently with probability 1/2) is almost surely isomorphic to R.

We write $x \sim y$ to denote that the vertices x and y are adjacent.

Alice's restaurant property: A graph Γ has this property if for every disjoint finite sets U and V of vertices there exists w such that $w \sim u$ for all $u \in U$ and $w \not\sim v$ for all $v \in V$.

With probability 1 a countable random graph has the Alice's restaurant property.

Any two countable graphs with the Alice's restaurant property are isomorphic (and such graphs exist).

Endomorphisms of R

A function $f : R \longrightarrow R$ is an **endomorphism** of R if

$$x \sim y \quad \text{implies} \quad (x)f \sim (y)f.$$

The monoid $\mathrm{End}(R)$ of endomorphisms of R has some interesting properties:

- (Bonato-Delić-Dolinka) $\mathbb{N}^{\mathbb{N}}$ embeds into $\mathrm{End}(R)$
- (Dolinka-Delić) $\mathrm{End}(R)$ has uncountably many ideals
- (Bonato-Delić) $\mathrm{End}(R)$ has $2^{\aleph_0}$ primitive idempotents.
- (Dolinka-Grey-MacPhee-Mitchell-Quick) every countable group is a maximal subgroup of $\mathrm{End}(R)$ in $2^{\aleph_0}$ distinct $\mathcal{D}$-classes.

A construction...

A construction: Let Γ be any countable graph. Then define Γ^* to be the graph with Γ as a subgraph and extra vertices

$$\{\, a_F \ :\ F \text{ is a finite subset of } \Gamma \,\}$$

and extra edges $u \sim v_F$ if $u \in F$.

Starting with $\Gamma_0 = \Gamma$, $\Gamma_1 = \Gamma_0^*$, $\ldots$, $\Gamma_{i+1} = \Gamma_i^*$,$\ldots$, we can show that:

$$R = \bigcup_{i=0}^{\infty} \Gamma_i.$$

Note that if $f : \Gamma_0 \longrightarrow R$ is any homomorphism, then $a_F \mapsto a_{(F)f}$ extends f to an endomorphism of R.

Sierpiński rank of endomorphisms of R

Theorem (Mitchell-Péresse-Hyde '13)

The Sierpiński rank of $\text{End}(R)$ *is* 2.

Proof. Let $R = \bigcup_{i=0}^{\infty} \Gamma_i$ where $\Gamma_0 = \bigcup_{i=0}^{\infty} \Gamma_{0,i}$ and every $\Gamma_{0,i} = R$.

- let $f_1, f_2, \ldots \in \text{End}(R)$
- $a : R \longrightarrow \Gamma_{0,0}$ be an isomorphism
  $b : \Gamma_{0,i} \longrightarrow \Gamma_{0,i+1}$ be an isomorphism
  $c : \Gamma_{0,n} \longrightarrow R$ be defined by $(x)c = (x)b^{-n}a^{-1}f_n$
- both b and c can be extended to endomorphisms of R
- So, $(x)ab^n \in \Gamma_{0,n}$ and so $(x)ab^nc = (x)ab^nb^{-n}a^{-1}f_n = (x)f_n$
- Again if we set $f_0 = b$, then $b = f_0 = ac$ and so $f_n = a(ac)^nc$. $\square$

Sierpiński rank of endomorphisms of other Fraïsse limits

Igor Dolinka, ‘The Bergman property for endomorphism monoids of some Fraïssé limits’, Forum Mathematicum to appear.

Igor generalized our theorem about the random graph to a wide class of countably infinite ultrahomogeneous structures.

The endomorphism monoids of the Fraïsse limits of the following classes have Sierpiński rank 2:

- finite posets;
- finite semilattices;
- finite distributive lattices;
- finite Boolean algebras.

Order automorphisms of the rationals

Theorem (Hyde-Jonušas-M-Péresse '13)

The (semigroup) Sierpiński rank of $\mathrm{Aut}(\mathbb{Q})$ *is* 2.

Sketch of the proof (that the Sierpiński rank ≤ 8).

Let $f_0, f_1, \ldots \in \mathrm{Aut}(\mathbb{Q})$. In each of the following cases, there exist $a, b, c, d \in \mathrm{Aut}(\mathbb{Q})$ such that:

- if $\mathrm{supp}(f_n) \subseteq [0, 1]$, then $f_n = [a^{b^n}, a^{b^{-n}c}]$
- if $|(x)f_n - x| \leq 1$, then $f_n = [a^{b^{2n}}, a^{b^{-2n}c}]^d[a^{b^{2n-1}}, a^{b^{-(2n-1)}c}]$
- if $|(x)f_n - x| \leq n$, then

$$f_n = \prod_{m=\frac{(n-1)n}{2}+1}^{\frac{n(n+1)}{2}} [a^{b^{2m}}, a^{b^{-2m}c}]^d[a^{b^{2m-1}}, a^{b^{-(2m-1)}c}]$$

- if the f_n are arbitrary, then the above equality holds. □

Open problems

1. What is the Sierpiński rank of $\mathrm{Aut}(R)$, $\mathrm{End}(\mathbb{Q})$, the automorphism groups and endomorphism semigroups of other Fraïssé limits?
2. If G is a group has group BP, then does G have semigroup BP? All known examples of groups with group BP also have semigroup BP.
3. Does there exist a semigroup S and $T \leq S$ with $|S \setminus T| < \infty$ but where S has BP but T does not?