Cayley Automaton Semigroups

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Definitions

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An *automaton* is a triple $\mathcal{A} = (Q, B, \delta)$ where:

- Q is a finite set of *states*
- B is a finite *alphabet*
- $\delta : Q \times B \rightarrow Q \times B$ is the *transition function.*

Definitions cont.

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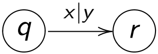

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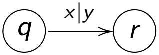

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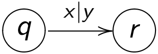

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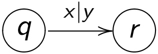

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Starting in state q and reading α gives an endomorphism of the $|B|$-ary rooted tree. Extending this to several states gives a homomorphism $\phi : Q^+ \to End(B^*)$.

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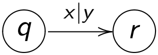

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We say that $\Sigma(\mathcal{A}) \cong im(\phi)$ is the *automaton semigroup.*

Cayley Automaton Semigroups

$\mathcal{C}(S)$ is the automaton arising from the Cayley Table of S. Each element $s \in S$ gives a state $\overline{s}$. Transitions are defined by right-multiplication in S: reading symbol t in state $\overline{s}$ moves us to state $\overline{st}$ and outputs symbol st.

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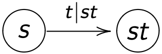

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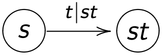

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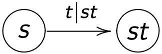

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Let $x \in S, \alpha \in S^*, \overline{q_i} \in \overline{S}$. Then

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For $\alpha = \alpha_1 \alpha_2 \ldots \alpha_n$ we have

$$\begin{aligned} \overline{q} \cdot \alpha &= (q\alpha_1)(\overline{q\alpha_1} \cdot \alpha_2 \ldots \alpha_n) \\ &= (q\alpha_1)(q\alpha_1\alpha_2)(\overline{q\alpha_1\alpha_2} \cdot \alpha_3 \ldots \alpha_2) \\ &\vdots \\ &= (q\alpha_1)(q\alpha_1\alpha_2) \ldots (q\alpha_1 \ldots \alpha_n) \end{aligned}$$

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So we can think of $\overline{q}$ as a function
$\overline{q} : \alpha_1\alpha_2 \dots \alpha_n \mapsto (q\alpha_1)(q\alpha_1\alpha_2) \dots (q\alpha_1 \dots \alpha_n)$.

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- (Silva and Steinberg 2005) Let G be a non-trivial finite group. Then $\Sigma(\mathcal{C}(G)) \cong F_{|G|}$
- (Mintz 2009) Let $T \leq S$. The $\Sigma(\mathcal{C}(T))$ divides $\Sigma(\mathcal{C}(S))$. If T is a non-trivial group then $\Sigma(\mathcal{C}(T)) \leq \Sigma(\mathcal{C}(S))$.

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Consider R_n. Then $\overline{x} \cdot \alpha = (x\alpha_1)(x\alpha_1\alpha_2)\ldots(x\alpha_1\ldots\alpha_n) = \alpha_1\alpha_2\ldots\alpha_n$ and $\overline{y} \cdot \alpha = \alpha_1\alpha_2\ldots\alpha_n$. So $\overline{x} = \overline{y}$ but $x \neq y$.

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Lemma

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($\Leftarrow$) Let $xa = ya$. Then

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Now let $w_1, \ldots, w_{n-1}$ be such that $w_1 w_2 \ldots w_{n-1} \neq 0$. Then $\overline{w_1} \cdot \ldots \cdot \overline{w_{n-2}} \cdot w_{n-1} = (w_1 w_2 \ldots w_{n-2} w_{n-1}) \neq 0^\omega$. Hence $\overline{w_1} \cdot \ldots \cdot \overline{w_{n-2}} \neq \overline{0}$. So $\Sigma(\mathcal{C}(S))$ is nilpotent of class $n-1$. □

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Lemma (Maltcev 2008)

Let S be finite. Then $\Sigma(\mathcal{C}(S))$ is free if and only if the minimal ideal K of S consists of a single $\mathcal{R}$-class in which every $\mathcal{H}$-class is non-trivial and there exists k such that $st = skt$ for all $s, t \in S$.

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- $L_n \cup B$ where L_n acts trivially on the band B
- If S is regular and self-automaton then it is a band

Self-Automaton Semigroups cont

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Theorem (M 2012)

Let B be a band. Then $B \cong \Sigma(C(B))$ under the map $b \mapsto \overline{b}$ if and only if the left-regular representation of B is faithful.

Self-Automaton Semigroups cont

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The semigroup defined by the following Cayley Table is not a band but is self-automaton:

	a	b	c	d
a	b	b	b	c
b	b	b	b	b
c	c	c	c	c
d	d	d	d	d

An alternative construction

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$$\alpha \cdot \overline{x} = (x\alpha_1)(x\alpha_1\alpha_2)(x\alpha_1\alpha_2\alpha_3)\dots$$

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$$\alpha \cdot (\overline{x_1} \cdot \overline{x_2}) = (\alpha \cdot \overline{x_1}) \cdot \overline{x_2}.$$

Denote the semigroup generated by the states with this right action by $\Pi(\mathcal{C}(S))$.

An alternative construction cont

Cain conjectures the following:

Conjecture

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Theorem

$S \cong \Pi(\mathcal{C}(S))$ *if and only if* S *is self-dual and* $S \cong \Sigma(\mathcal{C}(S))$.

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A complete classification of self-automaton semigroups (both self-dual and otherwise) remains an open question.

Thanks for listening!