Supernilpotence prevents dualizability

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FWF Der Wissenschaftsfonds.

What is a natural duality?

General idea (cf. Clark, Davey, 1998):

1. A duality is a correspondence between a category of algebras and a category of relational structures with topology.
2. **Representation:** Elements of the algebras are represented as continuous, structure preserving maps.
3. Classical example: **Stone duality** between Boolean algebras and Boolean spaces (totally disconnected, compact, Hausdorff)
4. Application, e.g., completions of lattices

For a finite algebra $\mathbf{A} = \langle A, F\rangle$, let $\underset{\sim}{\mathbf{A}} = \langle A, \mathcal{R}, \tau_d\rangle$ be an **alter ego**.

- $\mathcal{R} \subseteq \bigcup_{n\in\mathbb{N}}\{B \leq \mathbf{A}^n\} =: \mathrm{Inv}(\mathbf{A})$
- τ_d ... discrete topology on A

algebras

relational topological structures

$\mathbf{A}$ is **dualized** by $\underset{\sim}{\mathbf{A}}$ if $\forall \mathbf{B} \in ISP(\mathbf{A})$:

$$ED(\mathbf{B}) = \{e_b\colon \mathrm{Hom}(\mathbf{B}, \mathbf{A}) \to A, h \mapsto h(b) \mid b \in B\}$$

"Every morphism from $D(\mathbf{B})$ to $\underset{\sim}{\mathbf{A}}$ is an evaluation."

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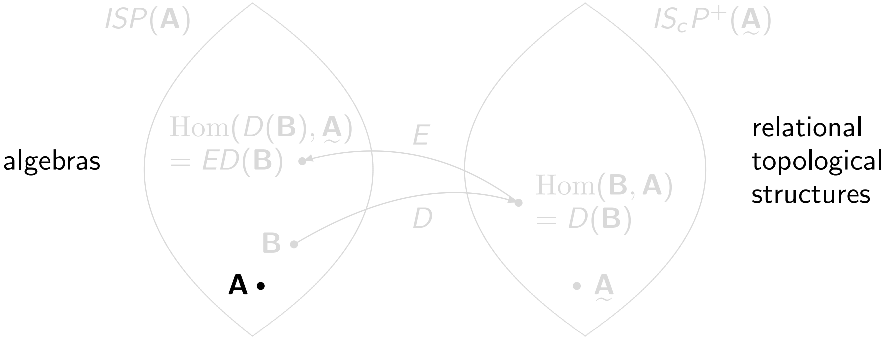

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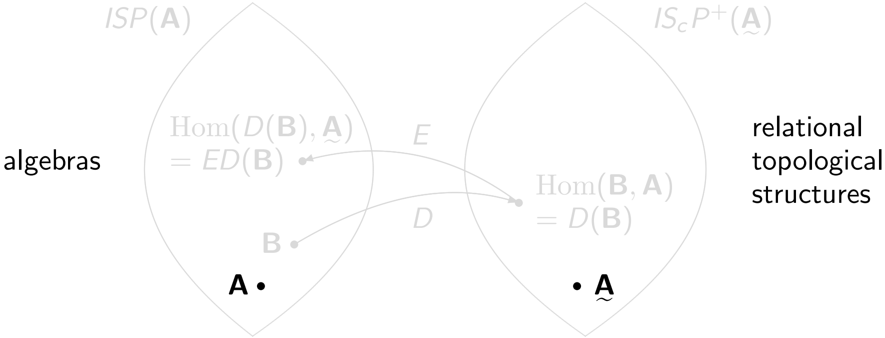

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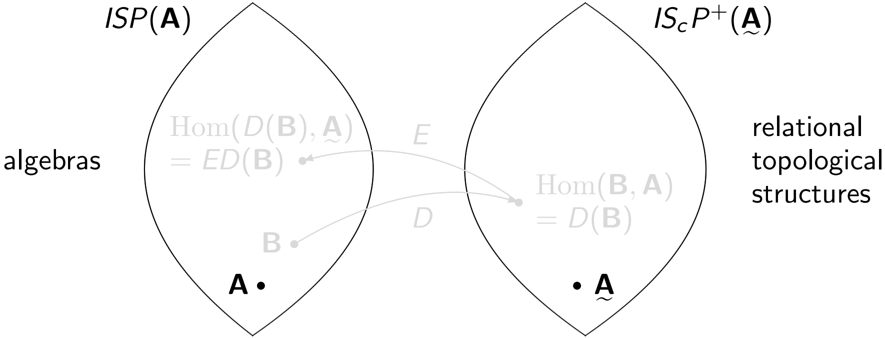

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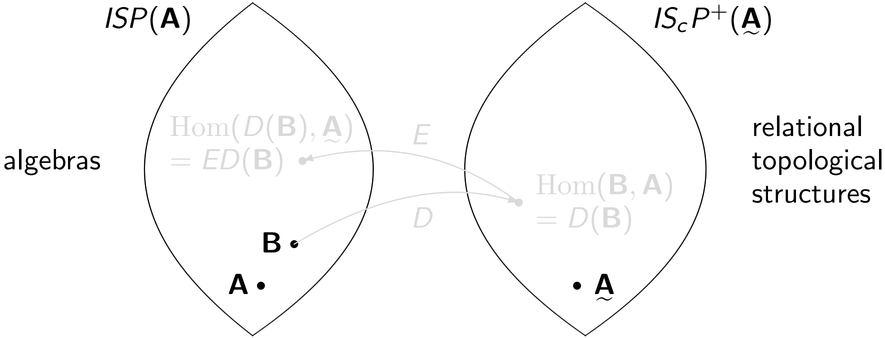

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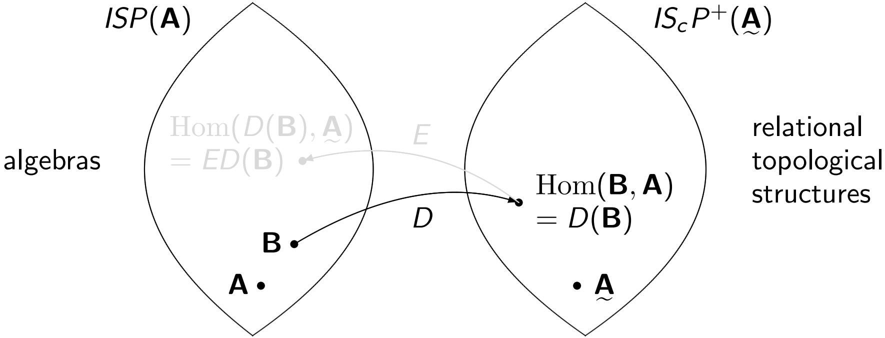

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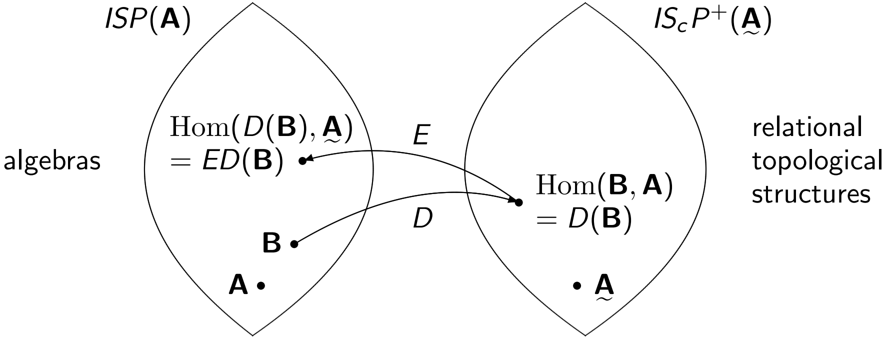

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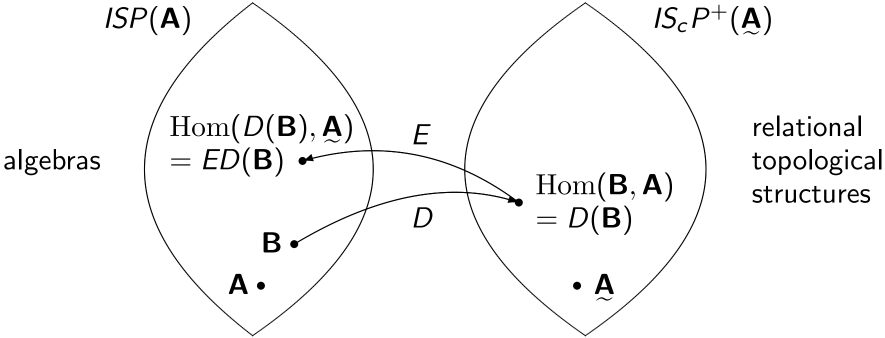

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When can $\mathbf{A}$ be dualized by some $\underset{\sim}{\mathbf{A}}$?

A is **not dualizable** iff $\exists \mathbf{B} \leq \mathbf{A}^S$ and a morphism α from $D(\mathbf{B}) \leq \underset{\sim}{\mathbf{A}}^B$ to $\underset{\sim}{\mathbf{A}} := \langle A, \mathrm{Inv}(\mathbf{A}), \tau_d \rangle$ that is not an evaluation.

Theorem (Davey, Heindorf, McKenzie, 1995)

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Problem (Clark, Davey, 1998)

Characterize dualizable (Mal'cev) algebras.

Theorem

1. A finite group is dualizable iff its Sylow subgroups are abelian. ($\Rightarrow$ Quackenbush, Szabó, 2002, $\Leftarrow$ Nickodemus, 2007)
2. A finite commutative ring with 1 is dualizable iff $J^2 = 0$. ($\Rightarrow$ Clark, Idziak, Sabourin, Szabó, Willard, 2001, $\Leftarrow$ Kearnes, Szendrei)
3. A finite ring (without 1) is dualizable only if $S^2 = 0$ for all nilpotent subrings S. (Szabó, 1999)

Our goal

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Our main result

A Mal'cev algebra **A** is **supernilpotent** if $[1_A, \ldots, 1_A] = 0_A$ for some higher commutator (Bulatov, 2001; Aichinger, Mudrinski, 2010). Equivalently **A** is pol. equivalent to a **direct product of nilpotent algebras of prime power order and finite type** (Freese, McKenzie).

Theorem (Bentz, M, submitted 2012)

Finite non-abelian supernilpotent Mal'cev algebras are (inherently) non-dualizable.

This yields the non-dualizability results from the previous slide because supernilpotence = nilpotence for groups and rings.

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How to show that $\mathbf{A}$ is not dualizable

Construct $\mathbf{B} \leq \mathbf{A}^S$ and $\alpha\colon \mathrm{Hom}(\mathbf{B},\mathbf{A}) \to A$, continuous and $\mathrm{Inv}(\mathbf{A})$-preserving such that α is not an evaluation by any element in $b \in B$.

Lemma (Ghost element method)

Let $\mathbf{A}$ be finite, $\mathbf{B} \leq \mathbf{A}^S$ and $B_0 \subseteq B$ infinite such that
$\exists N \; \forall h \in \mathrm{Hom}(\mathbf{B},\mathbf{A}) \; \exists b_h \in B_0$:

$$h(c) = h(b_h) \text{ for all but at most } N \text{ elements } c \in B_0.$$

1. Then $\alpha\colon \mathrm{Hom}(\mathbf{B},\mathbf{A}) \to A, h \mapsto h(b_h)$ is continuous, preserves $\mathrm{Inv}(\mathbf{A})$. α is an evaluation on every finite subset of $\mathrm{Hom}(\mathbf{B},\mathbf{A})$.
2. If α is an evaluation at $g \in A^S$, then $g_s = \alpha(\pi_s) = \pi_s(b_{\pi_s}) \; \forall s \in S$.
3. If $g \notin B$ (is a **ghost**), then $\mathbf{A}$ is not dualizable.

$\mathbf{A} := \langle \mathbb{Z}_4, +, 2x_1x_2 \rangle$ is not dualizable

Proof by ghost element method (adapted from Szabó ...):

Consider $\mathbf{B} \leq (\mathbf{A}^3)^{\mathbb{Z}}$ generated by

$$d_i := (\ldots, \bar{0}, \underset{i}{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \underbrace{\bar{0}, \ldots, \bar{0}}_{9}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \bar{0}, \ldots) \qquad (i \in \mathbb{Z}).$$

Then

$$2d_i d_{i-7} = (\ldots, \bar{0}, \underset{i}{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}, \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \bar{0}, \ldots) \in B$$

and

$$v_{ij} := (\ldots, \bar{0}, \underset{i}{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}, \bar{0}, \ldots, \bar{0}, \underset{j}{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}}, \bar{0}, \ldots) \in B \qquad (i < j).$$

$\mathbf{A} := \langle \mathbb{Z}_4, +, 2x_1x_2 \rangle$ is not dualizable, continued

Let $B_0 = \{v_{0j} \mid j \in \mathbb{N}\}$.

1. Then every $h\colon \mathbf{B} \to \mathbf{A}$ maps all but at most 56 elements of B_0 to the same image (Uses explicit construction of v_{0j} by the generators d_i).
2. The ghost

$$g := \left(\ldots, \bar{0}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \bar{0}, \ldots\right)$$

is not in B (By a parity argument that uses the explicit description of term operations).

Hence $\mathbf{A}$ is not dualizable.

Remark

This approach can be generalized from $\langle \mathbb{Z}_4, +, 2x_1x_2 \rangle$ to arbitrary supernilpotent Mal'cev algebras.

Nilpotence alone is not an obstacle

Theorem (Bentz, M, submitted 2012)

$\mathbf{A} := \langle \mathbb{Z}_4, +, 1, \{2x_1 \cdots x_k \mid k \in \mathbb{N}\}\rangle$ is nilpotent and dualized by $\underset{\sim}{\mathbf{A}} := \langle \mathbb{Z}_4, \{R \leq \mathbf{A}^4\}, \tau_d\rangle$.

Fun fact

All reducts

$$\langle \mathbb{Z}_4, +, 2x_1x_2, \ldots, 2x_1 \cdots x_k\rangle \qquad (k \in \mathbb{N})$$

of finite type are supernilpotent, hence non-dualizable.

Duality via partial clones

Partial operations on "conjunct-atomic definable" domains

$\mathrm{Clo}(\mathbf{A}) \ldots$ term operations on $\mathbf{A}$

$\mathrm{Clo}_{\mathrm{cad}}(\mathbf{A}) := \{f|_D \colon f \in \mathrm{Clo}(\mathbf{A}), D \underbrace{\text{is solution set of term identities on } \mathbf{A}}_{\mathrm{cad}}\}$

For $D \subseteq A^k$, a partial op $f : D \to A$ **preserves** a relation $R \subseteq A^n$ if

$$\forall r_1, \ldots, r_k \in R : f(r_1, \ldots, r_k) \in R \text{ whenever defined.}$$

Lemma (Davey, Pitkethly, Willard, 2012)

Assume $\mathbf{A}$ and $\mathcal{R} \subseteq \mathrm{Inv}(\mathbf{A})$ are **finite** such that $\mathrm{Clo}_{\mathrm{cad}}(\mathbf{A})$ is the set of all $\mathcal{R}$-preserving operations with cad domains over $\mathbf{A}$.
Then $\mathbf{A}$ is dualized by $\underset{\sim}{\mathbf{A}} := \langle A, \mathcal{R}, \tau_d \rangle$.

Follows from Third Duality Theorem and Duality Compactness.

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Proof:

1. Solution sets $D \subseteq \mathbb{Z}_4^k$ of term identities can be explicitly described.
2. $\mathrm{Clo}_{\mathrm{cad}}(\mathbf{A})$ is determined by the unary term operations and the 4-ary commutator relations (just like $\mathrm{Clo}(\mathbf{A})$).

Open

Problem

Is every finite abelian algebra in a CM variety dualizable? Every finite ring module?

Problem

Let **A** be a finite Mal'cev algebra with a non-abelian supernilpotent congruence α, i.e., $[\alpha, \dots, \alpha] = 0$. Is **A** necessarily non-dualizable?

Yes, if **A** is nilpotent.

Supernilpotence is not the only obstacle for dualizability

$\langle S_3, \cdot, \text{all constants} \rangle$ is not dualizable (Idziak, unpublished) but all its (super)nilpotent congruences are abelian.

Wild guess

A finite nilpotent **A** is dualizable iff all supernilpotent algebras in $HSP(\mathbf{A})$ are abelian.