Quasivarietes of symmetric, idempotent and entropic groupoids

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The 4th Novi Sad Algebraic Conference
Novi Sad 5-9.06.2013

Definition

A symmetric, idempotent and entropic groupoid $(G, \cdot)$ is an algebra satisfying the identities

$$(x \cdot y) \cdot y = x, \tag{S}$$

$$x \cdot x = x, \tag{I}$$

$$(x \cdot y) \cdot (z \cdot t) = (x \cdot z) \cdot (y \cdot t). \tag{E}$$

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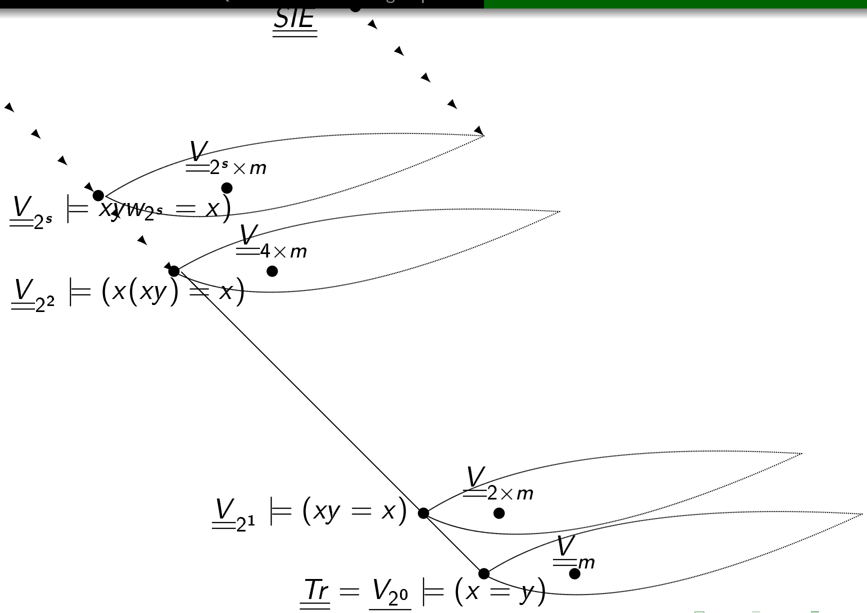
Theorem B. Roszkowska-Lech

The lattice $\mathcal{L}(\underline{\underline{SIE}})$ *of all the subvarieties of the variety* $\underline{\underline{SIE}}$ *of symmetric, idempotent and entropic groupoids is isomorphic to the lattice* $(\mathbb{N} \cup \{\infty\}, |)$ *of positive integers ordered by the divisibility relation with the greatest element* ∞.


$\underline{\underline{SIE}}$
$\underline{\underline{V}}_{2^s \times m}$
$\underline{\underline{V}}_{2^s} \models (xyw_{2^s} = x)$
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$\underline{\underline{V}}_{m}$
$\underline{\underline{Tr}} = \underline{\underline{V}}_{2^0} \models (x = y)$

A *SIE*-groupoid $(G, \cdot)$ is **cancellative** if it satisfies the cancellation quasi-identities

$$(Cl) \qquad \begin{cases} (xy = xz) \Rightarrow (y = z) \\ (yx = zx) \Rightarrow (y = z) \end{cases}$$

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Denote by $\mathrm{Q}(\alpha)$ the quasivariety of *SIE*-groupoids defined by the quasi-identity α. Let $\mathbb{Z}_2$ be the two-element left zero band with elements $0, 1$.
Denote by $\mathrm{N}(\mathbb{Z}_2)$ the class of *SIE*-groupoids with no subalgebra isomorphic to $\mathbb{Z}_2$.

$$(\alpha) \quad x \cdot y = x \Rightarrow x = y.$$

Lemma

The following two classes coincide

$$\mathsf{Q}(\alpha) = \mathsf{N}(\mathbb{Z}_2).$$

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Theorem

The class $\underline{\underline{SIE}}_{cl}$ of cancellative SIE-groupoids is a subquasivariety of the variety $\underline{\underline{SIE}}$ of symmetric, idempotent and entropic groupoids satisfying the quasi-identity:

$$(\alpha) \quad x \cdot y = x \Rightarrow x = y.$$

Belkin's construction of the lattice $K(\omega)$ for the cardinal ω.

Let ω^+ denote $\omega \cup \{\infty\}$. Let $K(\omega)$ be the set of functions

$$f : \omega^+ \to \omega^+,$$

where $f(\infty) \in \{0, \infty\}$ and $f(\infty) = 0$ implies that $f(\omega) \subseteq \omega$ and $f(i) = 0$ for almost all $i \in \omega$. Then $K(\omega)$ is a distributive lattice with respect to the following operations:

$$(f \vee g)(i) = max\{f(i), g(i)\}, \; (f \wedge g)(i) = min\{f(i), g(i)\},$$

where $i < \infty$ for all $i \in \omega$.

Theorem

The lattice $\mathcal{L}_q(\underline{\underline{SIE}}_{cl})$ of quasivarieties of cancellative symmetric, idempotent and entropic groupoids is isomorphic to the lattice $K(\omega)$:

$$\mathcal{L}_q(\underline{\underline{SIE}}_{cl}) \cong K(\omega).$$

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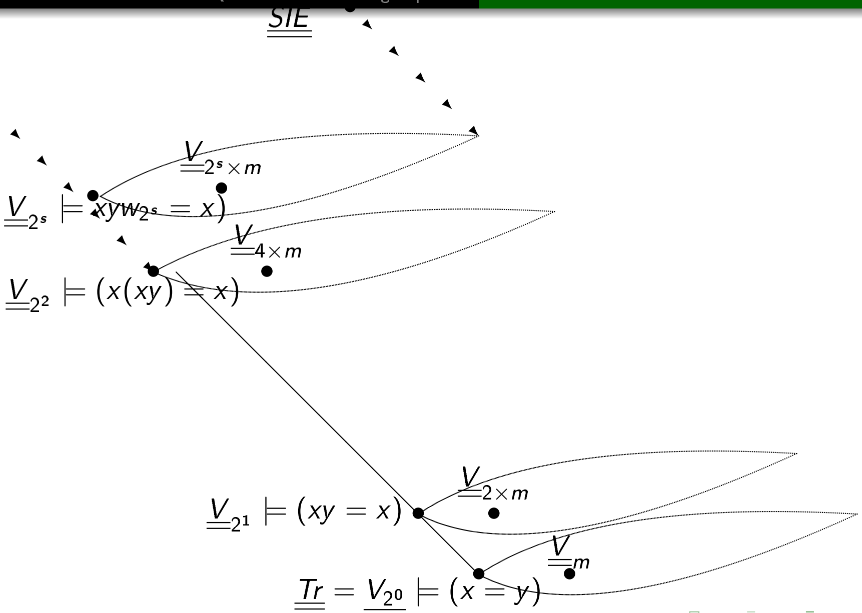
The quasivariety $\underline{\underline{Q}}(\mathbb{Z})$ is a **minimal** quasivariety of the variety $\underline{\underline{SIE}}$ and a **minimal** quasivariety of the variety $\underline{\underline{SIE}}_{cl}$. It is the minimal quasivariety not contained in any minimal variety.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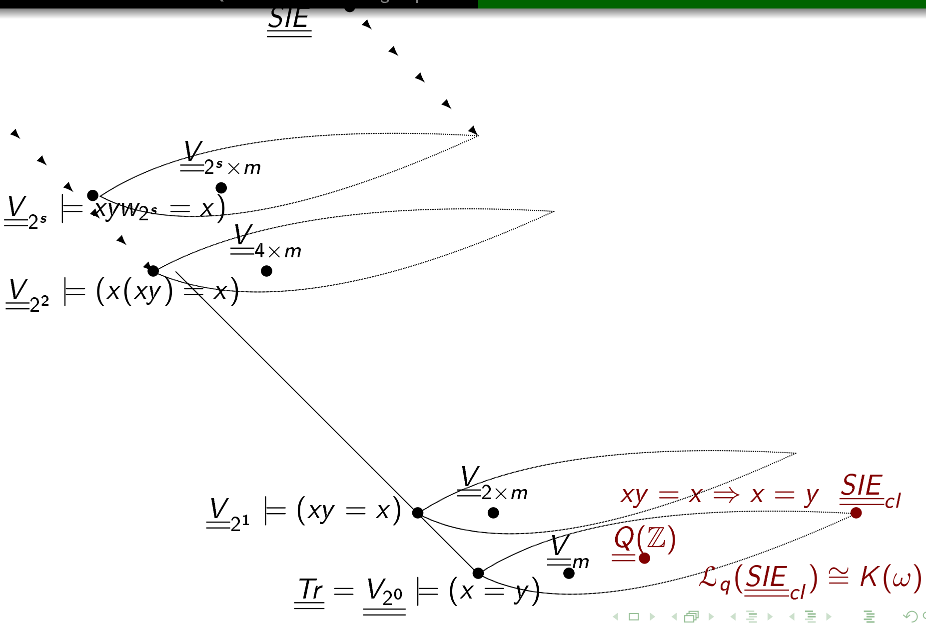
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Theorem L.Hogben and C.Bergman

Let $\underline{\underline{V}}$ be residually finite and of finite type, or residually and locally finite. Then $\underline{\underline{V}}$ is deductive if and only if every subdirectly irreducible algebra in $\underline{\underline{V}}$ is primitive.

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An algebra $\mathbf{P} \in \underline{\underline{V}}$ is **primitive** iff $\mathbf{P}$ is finite, subdirectly irreducible and, for all $\mathbf{A} \in \underline{\underline{V}}$, $\mathbf{P} \in H(\mathbf{A}) \Rightarrow \mathbf{P} \in IS(\mathbf{A})$.

Lemma

A variety $\underline{\underline{V}}_m$ is deductive for any odd natural numer m.

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Theorem

Let m be an odd natural number and s a natural number. Then the variety $\underline{\underline{V}}_{2^s m}$ is deductive iff $s = 0$ or $s = 1$.

A variety $\underline{\underline{V}}_{2^s m}$ for an odd natural number m and a natural number $s > 1$ is not deductive.

Subdirectly irreducible *SIE*-grupoids in $\underline{\underline{V}}_4$ were described by J.Plonka. They are two subdirectly irreducible groupoids in $\underline{\underline{V}}_4$. There are $\mathbf{P}_1^2 = (\{0, 1, , d\}, \cdot)$ and $\mathbf{P}_2^2 = (\{0, 1, a, b\}, \cdot)$ with operations defined as follows:

$\mathbf{P}_1^2 =$

$\cdot$	0	1	d
0	0	0	1
1	1	1	0
d	d	d	d

$\mathbf{P}_2^2 =$

$\cdot$	0	1	a	b
0	0	0	0	1
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a	a	a	a	a
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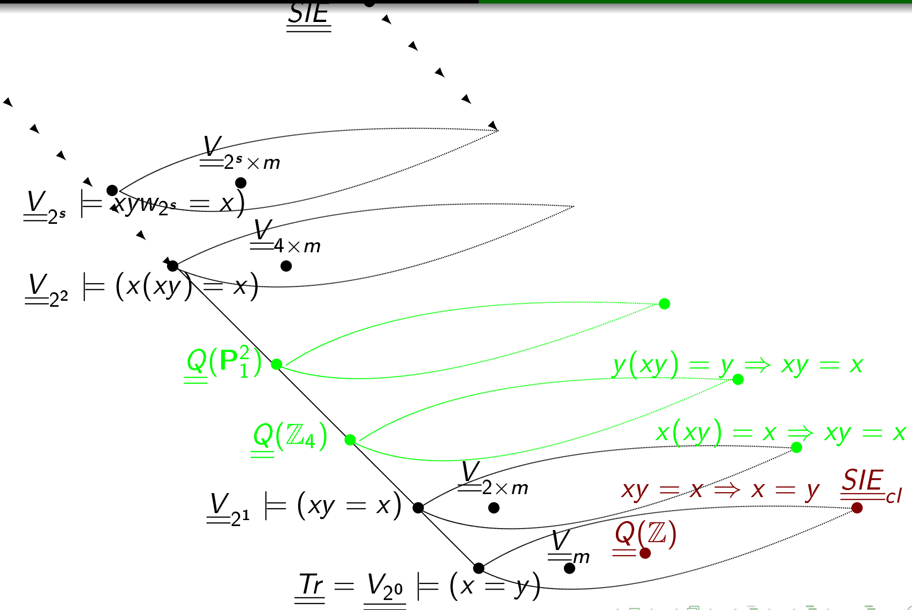
Theorem

The following quasivarieties form a strictly increasing chain:

$$\underline{\underline{V}}_2 \lneq \underline{\underline{Q}}(\mathbb{Z}_4) \lneq \underline{\underline{Q}}(\mathbf{P}_1^2) \lneq \underline{\underline{V}}_4$$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V 4×m
V 2^2 ⊨ (x(xy) = x)
Q(P^2_1)
y(xy) = y ⇒ xy = x
Q(Z_4)
x(xy) = x ⇒ xy = x
V 2×m
xy = x ⇒ x = y
SIE_cl
V 2^1 ⊨ (xy = x)
Q(Z)
V m
Tr = V 2^0 ⊨ (x = y)

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$$\underline{\underline{Q}}(\mathbb{Z}_2) = \underline{\underline{V}}_2 \lneq \underline{\underline{Q}}(\mathbb{Z}_{2^2}) \lneq \ldots \lneq \underline{\underline{Q}}(\mathbb{Z}_{2^s}) \lneq \ldots$$

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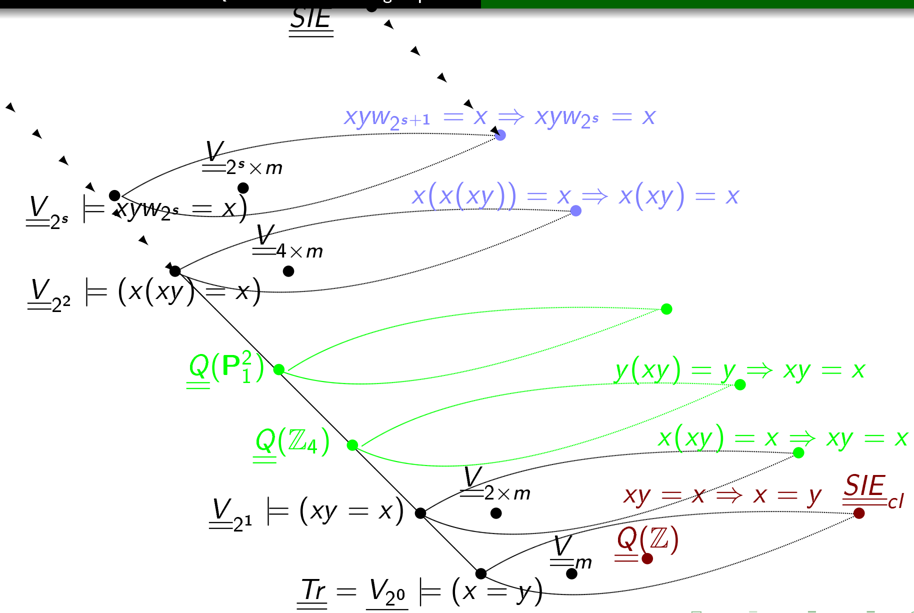
where $\mathbf{P}_1^{2^s}$ *is a subdirectly irreducible groupoid in* $\underline{\underline{V}}_{2^s}$ *and* $\mathbf{P}_1^{2^s} \notin \underline{\underline{V}}_{2^{s+1}}$*, for natural number* $s \geq 1$*.*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Thank you for your attention.